# MECHANICS

*for*

# M.Sc. Entrance

IIT-JAM | JNU | TIFR | BHU | ISM | JNCASR | DU

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## MECHANICS

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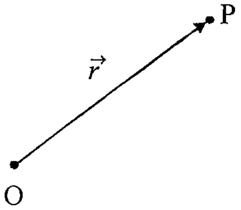
# Chapter 1

# *Velocity and Accceleration in Different Coordinate system*

In physics basic laws are first introduced for a point partile and then laws are extended to system of particles or continuous bodies. Therefore, we also begin the discussion with point particle and later on we will study collection of particles or rigid body.

To write equations governing the dynamics of a aprticle we need its position vectors, velocity, acceleration etc. Therefore we first introduce these elementaty concepts.



**Position vector:**
It is a vector directed from some point to location of particle. In the figure shown $\vec{r}$ is position vector of particle 'P' with respect to point 'O'. If we specify the coordinate of particle then position vector can be expressed in terms of coordinates and unit vectors used in that coordinate system.

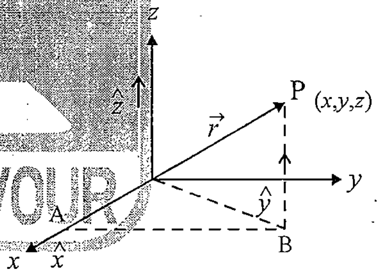
**In cartesian coordinate system:**
Coordinates of particle are written as $(x, y, z)$ and unit vectors along $x, y, z$ axes are $\hat{x}, \hat{y}$, and $\hat{z}$ respectively. Therefoer, from figure,

$$OA = x,\ AB = y,\ BP = z$$

and $$\vec{OP} = \vec{OB} + \vec{BP}$$

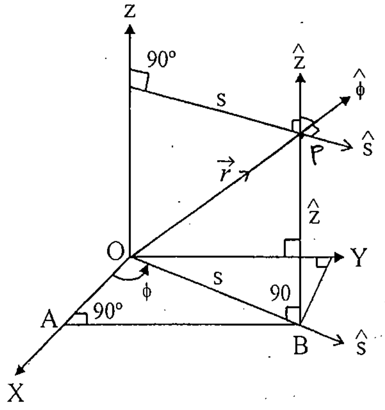
$$\vec{r} = \vec{OA} + \vec{AB} + \vec{BP} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\therefore \quad \boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$$



Unit vectors are taken in the directions in which coordinates increase.

**In cylindrical system** coordinates of particle are written as $(s, \phi, z)$ and unit vectors along the increasing direction of coordinates are $(\hat{s}, \hat{\phi}, \hat{z})$. $s$ is perpendicular distance of particle from z-axis, $\phi$ is its angular position with respect to $x$-axis and $z$ is its distance above $x$-$y$ plane.



Therefore, from figure $\mathrm{OB} = s$, $\angle\mathrm{AOB} =$, $\mathrm{BP} = z$ and $\vec{\mathrm{OP}} = \vec{\mathrm{OB}} + \vec{\mathrm{BP}}$

$$\boxed{\vec{r} = s\hat{s} + z\hat{z}}$$

relation with cartesian coordinates

$$\boxed{x = s\cos\phi,\ y = s\sin\phi,\ z = z} \text{ and } \boxed{\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y},\ \hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y},\ z = \hat{z}}$$

**In spherical polar coordinates system,** coordinates of particle are written as $(r,\theta,\phi)$ and unit vector in increasing direction of coordinates are $(\hat{r},\hat{\theta} \text{ and } \hat{\phi})$. r is the distance of particle from origin, $\theta$ and $\phi$ are angular position with respect to $z$ and $x$ axes respectively.

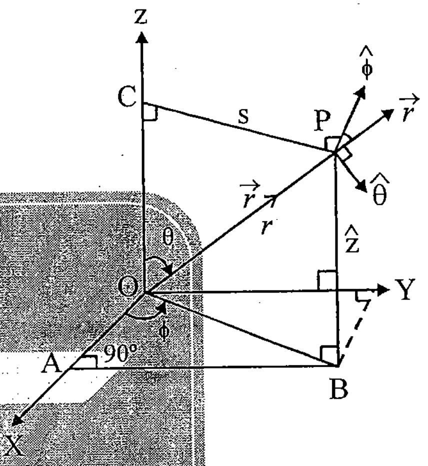
From figure $\mathrm{OP} = r$, $\angle\mathrm{COP} = \theta$, $\angle\mathrm{AOB} = \phi$ and $\vec{\mathrm{OP}} = r\hat{r}$

$$\therefore \quad \boxed{\vec{r} = r\hat{r}}$$

$\mathrm{OB} = \mathrm{CP} = r\sin\theta, \qquad \mathrm{BP} = r\cos\theta$

$\therefore \quad \mathrm{AB} = \mathrm{OB}\sin\phi = r\sin\theta.\sin\phi$ and

$OA = OB\cos\phi = r\sin\theta\cos\phi$



Relation with cartesian coordinate,

$$\boxed{x = r\sin\theta.\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta}$$

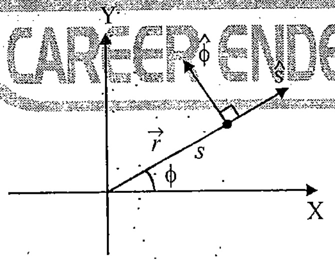
And $\quad \hat{r} = \sin\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) + \cos\theta\hat{z}$

$$\hat{\theta} = \cos\theta(\cos\phi\hat{x} + \sin\phi\hat{y}) - \sin\theta\hat{z}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

If motion of a particle is confined in one plane then only two coordinates are required to describe its position. We can either use cartesian coordinates $(x, y)$ or plane polar coordinates $(s, \phi)$.

Thus if a particle is moving on a plane then its position vector can be written as



$$\vec{r} = x\hat{x} + y\hat{y}$$

Or, $\quad \vec{r} = s\hat{s}$ in (plane polar coordinate)

Plane polar coordinates $(s,\phi)$ are the same coordinates which are used in cylindrical coordinates system.

Notice that, $\hat{x}, \hat{y}$ and $\hat{z}$ have a fixed direction as they are along the $x, y$ and $z$ axes, whereas $\hat{r}, \hat{s}, \hat{\theta}, \hat{\phi}$ etc do not have fixed directions. Therefore, $\hat{x}, \hat{y}, \hat{z}$ are constant unit vectors but $\hat{r}, \hat{s}, \hat{\theta}, \hat{\phi}$ are not constant unit vectors.

Thus, $\dfrac{d\hat{x}}{dt} = 0, \dfrac{d\hat{y}}{dt} = 0, \dfrac{d\hat{z}}{dt} = 0$ and $\dfrac{d\hat{r}}{dt} \neq 0, \dfrac{d\hat{s}}{dt} \neq 0, \dfrac{d\hat{\theta}}{dt} \neq 0, \dfrac{d\hat{\phi}}{dt} \neq 0$

Derivative of unit vectors $(\hat{r}, \hat{\theta}, \hat{s}, \hat{\phi})$ can easily be found by using their relation with $(\hat{x}, \hat{y}, \hat{z})$.

**For example:** In plane polar or cylindrical coordinates, $\hat{s} = \cos\phi\hat{x} + \sin\phi\hat{y}$ and $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$

$$\therefore \qquad \frac{d\hat{s}}{dt} = -\sin\phi\frac{d\phi}{dt}\hat{x} + \cos\phi\frac{d\phi}{dt}\hat{y}$$

$$= \left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right)\frac{d\phi}{dt} = \hat{\phi}\dot{\phi}$$

and $$\frac{d\hat{\phi}}{dt} = -\cos\phi\frac{d\phi}{dt}\hat{x} - \sin\phi\frac{d\phi}{dt}\hat{y}$$

$$= -\left(\cos\phi\hat{x} + \sin\phi\hat{y}\right)\frac{d\phi}{dt} = -\hat{s}\dot{\phi}$$

$$\therefore \qquad \boxed{\frac{d\hat{s}}{dt} = \hat{\phi}\dot{\phi}, \qquad \frac{d\hat{\phi}}{dt} = -\hat{s}\dot{\phi}}$$

**Velocity:** Average velocity is defined as,

$$\vec{u}_{av} = \frac{\text{total displacement}}{\text{total time taken}} = \frac{\text{change in position vector}}{\text{total time taken}}$$

$$= \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

Instantaneous velocity (velocity at any instant of time) is defined as time derivative of position vector.

Instantaneous velocity, $\vec{v} = \frac{d\vec{r}}{dt}$.

By expressing $\vec{r}$ in different coordinate system.

**In cartesian coordinate system:**

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Therefore, $$\vec{v} = \frac{d}{dt}\left(x\hat{x} + y\hat{y} + z\hat{z}\right) = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$\boxed{\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}}$$

Or, $\quad v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$v_x = \dot{x} = \frac{dx}{dt}$ is component of velocity in $x$ direction.

$v_y = \dot{y} = \frac{dy}{dt}$ is component of velocity in y direction

$v_z = \dot{z} = \frac{dz}{dt}$ is component of velocity in z direction

**In plane coordinate system:**

$$\vec{r} = s\hat{s}$$

Therefore, $$\vec{v} = \frac{d}{dt}\left(s\hat{s}\right) = \frac{ds}{dt}\hat{s} + s\frac{d\hat{s}}{dt}$$

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}} \qquad \left\{\because \frac{d\hat{s}}{dt} = \dot{\phi}\hat{\phi}\right\}$$

Or, $\vec{v} = v_s\hat{s} + v_\phi\hat{\theta}$

$v_s = \dot{s} = \dfrac{ds}{dt}$ is component velocity in $\hat{s}$ direction and it is called radial velocity

$v_\phi = s\dot{\phi} = s\dfrac{d\phi}{dt}$ is component velocity in $\hat{\phi}$ direction and it is called transverse velocity.

**In cylindrical coordinate system:**

$$\vec{r} = s\hat{s} + z\hat{z}$$

Therefore, $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\left(s\hat{s} + z\hat{z}\right)$

$$= \frac{ds}{dt}\hat{s} + s\frac{d\hat{s}}{dt} + \frac{dz}{dt}\hat{z}$$

$$\boxed{\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}} \qquad \left\{\because \frac{d\hat{s}}{dt} = \dot{\phi}\hat{\phi}\right\}$$

**Acceleration :** Average acceleration is defined as

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

Instantaneous acceleration is defined as time derivative of velocity vector.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

By expressing $\vec{v}$ indifferent coordinatge system. We can writge acceleration of a particle in different coordinate system.

**In cartesian coordinate:**

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

Therefore, $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}\right)$

$$\boxed{\vec{a} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}}$$

or, $\vec{a} = \left(a_x\hat{x} + a_y\hat{y} + a_z\hat{z}\right)$

$a_x = \ddot{x} = \dfrac{d}{dt}\left(\dot{x}\right) = \dfrac{dv_x}{dt}$ is component of acceleration along $x$-direction.

$a_y = \ddot{y} = \dfrac{d}{dt}\left(\dot{y}\right) = \dfrac{dv_y}{dt}$ is component of acceleration along $y$-direction.

$a_z = \ddot{z} = \dfrac{d}{dt}\left(\dot{z}\right) = \dfrac{dv_z}{dt}$ is component of acceleration along $z$-direction.

Since, $a_x = \dfrac{dv_x}{dt}$, therefore, if velocity along $x$-direction is constant then acceleration along $x$-direction must be zero.

**In Plane polar coordinate :**

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi}$$

Therefore, $$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{s}\hat{s} + s\dot{\phi}\hat{\phi}\right)$$

$$= \ddot{s}\hat{s} + \dot{s}\frac{d\hat{s}}{dt} + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\frac{d\hat{\phi}}{dt}$$

$$= \ddot{s}\hat{s} + \dot{s}\left(\dot{\phi}\hat{\phi}\right) + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\left(-\dot{\phi}\hat{s}\right)$$

$$\boxed{\vec{a} = \left(\ddot{s} - s\dot{\phi}^2\right)\hat{s} + \left(2\dot{s}\dot{\phi} + s\ddot{\phi}\right)\hat{\phi}}$$

Or, $$\vec{a} = a_s\hat{s} + a_\phi\hat{\phi}$$

$a_s = \ddot{s} - s\dot{\phi}^2$ is component of acceleration along $\hat{s}$ and its called radial acceleration. Clearly, it is not equal to time derivative of radial component of velocity $\left(v_s = \dot{s}\right)$.

Therefore, if $v_s$ = constant , then $a_s$ may not be zero.

$a_\phi = 2\dot{s}\dot{\phi} + s\ddot{\phi}$ is component of acceleration in $\hat{\phi}$ direction. It is called transverse acceleration. Clearly it is also not equal to time derivative of radial component of velocity $\left(v_\phi = s\dot{\phi}\right)$

i.e. $$a_\phi \neq \frac{d}{dt}\left(v_\phi\right)$$

Therefore, if $v_\phi$ = constant then $a_\phi$ may not be zero.

**In cylindrical coordinate system:**

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

First two terms are same as in plane polar coordinate.

Therefore, $$\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{s} - s\dot{\phi}^2\right)\hat{s} + \left(2\dot{s}\dot{\phi} + s\ddot{\phi}\right)\hat{\phi} + \ddot{z}\hat{z}}$$

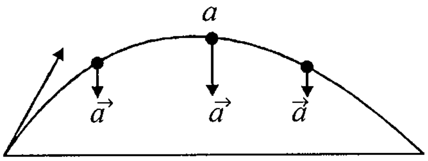
**How to choose a coordinate system:**

Generally speaking, dynamics of a particle can be studied using any coordinate system. However judicious selection of coordinate system makes task easy.

One dimensional cases is simple because we just need one coordinate which we can take either $x$ or $y$ or $z$.

However in two dimensional cases one has to be contious. In most of the two dimensional problems if acceleration of the particle is directed towards a point then plane polar coordinate is conveninet to use. However, if accleration is not directed towards a point then cartesian coordinate is right choice.

**For example :** A projectile motion in which particle is projected with a small speed at some angle with horizontal. It moves with constant acceleration which is directed downward. In this case acceleration is not directed towards a point. Therefore, we can conveniently use cartesian coordinates.



In planetary motion, acceleration of planet is always directed towards centre of the sun. Therefore in this case plane polar coordinate makes calculation simple.

If a particle is attached to a point with a string or spring then also generally polar coordinate is helpfull. For example, in case of simple pendulum.

**Formulae of kinematics (uniformly accelerated motions):** When a constant force acts on a particle its acceleration is constant. If $u_x, v_x$ be initial and final velocity of a particle along $x$-direction and $a_x$ be its acceleration along $x$ direction then.

$$\left.\begin{aligned} v_x &= u_x + a_x t \\ v_x^2 &= u_x^2 + 2a_x x \\ x &= u_x t + \tfrac{1}{2} a_x t^2 \end{aligned}\right\} \text{if } a_x = \text{constant}$$

where $x$ is displacement along $x$-direction and $t$ is time taken. We can write similar relations for $y$ and $z$ direction also.

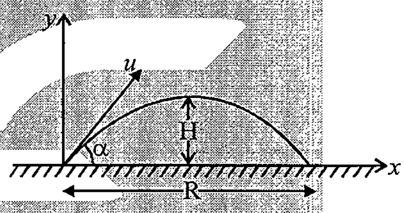
**Note:** If acceleration of particle is not constant then we cannot use formulae of kinematics. In that case we start with either definition of velocity or definition of acceleration i.e.

$$v_x = \frac{dx}{dt} \quad \text{or} \quad a_x = \frac{dv_x}{dt}$$

we may also have to use $a_x = \dfrac{F_x}{m}$ and $\dfrac{dv_x}{dt} = \dfrac{dv_x}{dx} \cdot \dfrac{dx}{dt} = \dfrac{v_x dv_x}{dx}$ as the case may be

**Projectile motion:** If $u$ be the projection speed and $\alpha$ be angle of projection then equation of path of projectile is,

$$y = x \tan\alpha - \frac{gx^2}{2u^2 \cos^2\alpha}$$



Range, $R = \dfrac{u^2 \sin 2\alpha}{g}$

Maximum height, $H = \dfrac{u^2 \sin^2\alpha}{2g}$

Total time of flight, $T = \dfrac{2u \sin\alpha}{g}$

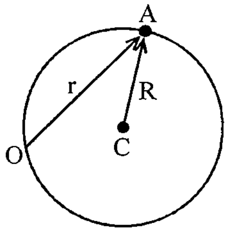
Coordinate of projectile at any time $t < T$

$$x = u\cos\alpha\, t, \; y = u \sin\alpha\, t - \frac{1}{2} g t^2$$

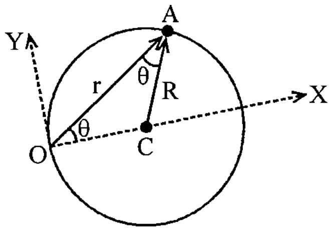
**Note:** Above results for projectile are applicable only if projectile is thrown from ground and it finally lands on ground. And projection speed is small so that projectile does not go too high. If projection speed is not small then height will be large and in that case we need to consider variation of acceleration due to gravity.

## SOLVED PROBLEMS

1. A particle $A$ moves along a circle of radius $R = 50$ cm so that its radius vector $r$ relative to point O (figure (a)) rotates with the constant angular velocity $\omega = 0.40$ rad/sec. Find the modulus of the velocity of the particle and modulus and direction of its total acceleration.



(a)



(b)

**Soln.** Consider $X$ and $Y$ axes as shown in (figure (b)). Using sine law in triangle $CAO$, we get $\dfrac{r}{\sin(\pi - 2\theta)} = \dfrac{R}{\sin\theta}$

or $$\frac{r}{2\sin\theta\cos\theta} = \frac{R}{\sin\theta}$$

$\therefore \quad r = 2R\cos\theta$

Now $\mathbf{r} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} = 2R\cos^2\theta\,\mathbf{i} + 2R\cos\theta\sin\theta\,\mathbf{j}$

Now $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -4R\cos\theta\sin\theta\dfrac{d\theta}{dt}\mathbf{i} + 2R\cos 2\theta\dfrac{d\theta}{dt}\mathbf{j} = -2R\sin 2\theta\,\omega\,\mathbf{i} + 2\theta\,\omega\,\mathbf{j}$

$\therefore \quad |\mathbf{v}| = 2\omega R$

Further

$$\mathbf{a} = \frac{dv}{dt} = 4R\cos 2\theta\,\omega\frac{d\theta}{dt}\mathbf{i} - 4R\omega\sin 2\theta\frac{d\theta}{dt}\mathbf{j} = -4R\omega^2\cos 2\theta\,\mathbf{i} - 4R\omega^2\sin 2\theta\,\mathbf{j}$$

$|\mathbf{a}| = 4R\omega^2.$

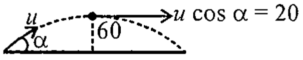
2. After being hit, a golf ball reaches a maximum height of 60m with speed of 20m/s. Right after being hit the speed of the ball is (take $g$ = 10 m/s$^2$) **[GP-2008]**

(a) 100m/s (b) 80m/s (c) 60m/s (d) 40m/s

**Soln.** (d) If $u$ be the speed just after being hit at angle $\alpha$ then

$u\cos\alpha = 20$ ... (i)

$$\frac{u^2\sin^2\alpha}{2g} = 60$$



$\therefore \quad u^2\sin^2\alpha = 1200$ ... (ii)

on squaring (i) and adding to (ii) we get

$u^2 = 1200 + 400$

$\therefore \quad u = 40$

3. A particle starts moving with acceleration $a = \alpha - \beta v$ where $\alpha$ and $\beta$ are constants and $v$ is its instantaneous speed. Find velocity of particle as a function of time and also find its terminal velocity.

**Soln:** Acceleration is variable therefore we start with definition of acceleration

$$a = \frac{dv}{dt} \text{ or } \alpha - \beta v = \frac{dv}{dt} \text{ or } \int_0^v \frac{dv}{\alpha - \beta v} = \int_0^t dt$$

$$-\frac{1}{\beta}\ln\left(\frac{\alpha - \beta v}{\alpha}\right) = t \text{ or, } 1 - \frac{\beta v}{\alpha} = e^{-\beta t} \qquad \therefore v = \frac{\alpha}{\beta}(1 - e^{-\beta t})$$

as $t \to \infty$, $v = \frac{\alpha}{\beta}$ = constant. This is terminal velocity.

4. A projectile is thrown at an angle $\alpha$ with horizontal with initial speed $u$ after what time the projectile's velocity will be perpendicular to its initial direction.

**Soln.** Suppose velocity after time $t$ becomes perpendicular to initial velocity. Velocity after time $t$ is

$$\vec{v}(t) = \dot{x}\hat{i} + \dot{y}\hat{j} \quad = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$$

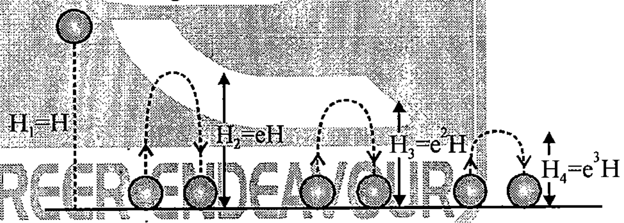
Initial velocity $\vec{u} = u(\cos\alpha\hat{i} + \sin\alpha\hat{j})$

$\vec{v}$ is $\perp$ to $\vec{u}$, therefore, $\vec{v}.\vec{u} = 0 \quad \therefore u\cos^2\alpha + u\sin^2\alpha - gt\sin\alpha = 0 \quad (\because u \neq 0)$

$$\therefore t = \frac{u}{g\sin\alpha}$$

5. A ball is dropped from a height H. It bounces back up to a height $e$ times after hitting the ground. If $e<1$, after what time the ball will finally come to rest.

**Soln.** Ball moves under the effect of gravity. Therefore magnitude of acceleration of ball during upward or downward movement remains constant.



For first bounce

$$y = H,\ u_y = 0,\ a_y = g$$

$$y = u_y t + \frac{1}{2}a_y t^2$$

$$\therefore H = 0 + \frac{1}{2}gt^2 \qquad \therefore t = \sqrt{\frac{2H}{g}}$$

Therefore time of fall before first bounce $t_1 = \sqrt{\frac{2H}{g}}$

Under gravity, time for going up a certain distance is equal to time for going down. Therefore time interval between first bounce and second bounce is

$$t_2 = 2\sqrt{\frac{2H_2}{g}} = 2\sqrt{\frac{2eH}{g}}$$

Similarly $t_3 = 2\sqrt{\frac{2e^2H}{g}}$, $t_4 = 2\sqrt{\frac{2e^3H}{g}}$,.....

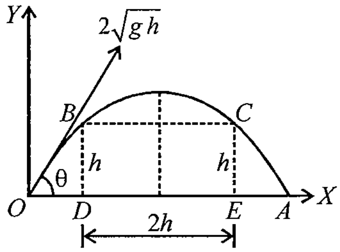
Therefore, total time elapsed before ball stops is

$$T = t_1 + t_2 + t_3 + ......$$

$$= \sqrt{\frac{2H}{g}}\left[1 + 2\left(e^{1/2} + e^{2/2} + e^{3/2} + ....\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2\frac{e^{1/2}}{1 - e^{1/2}}\right] = \sqrt{\frac{2H}{g}}\left(\frac{1 + \sqrt{e}}{1 - \sqrt{e}}\right)$$

6. A particle is projected with velocity $2\sqrt{(hg)}$ so that it just clears two walls of equal height $h$ which are at a distance $2h$ from each other. Show that the time of passing between the walls is $2\sqrt{(h/g)}$.

**Soln.** The situation is shown in figure.



Suppose the particle is at a height $h$ after time $t$. Then,

$$h = 2\sqrt{gh}\,\sin\theta\cdot t - \frac{1}{2}gt^2$$

or $$gt^2 - 4\sqrt{gh}\,\sin\theta\cdot t + 2h = 0 \qquad \text{... (i)}$$

This equation gives the two values of time at which the height of the particle is $h$. Let these times be $t_1$ and $t_2$ respectively. Then,

$$(t_1 - t_2)^2 = (t_1 + t_2) - 4t_1t_2$$

From equation (i)

$$t_1 + t_2 = \frac{4\sqrt{(gh)}\,\sin\theta}{g} \text{ and } t_1t_2 = \frac{2h}{g}$$

$$\therefore \quad (t_1 - t_2)^2 = \frac{16gh\sin^2\theta}{g^2} - \frac{8h}{g} = \frac{8h}{g}[2\sin^2\theta - 1]$$

or $$(t_1 - t_2) = \sqrt{\left(\frac{8h}{g}\right)}\,[2\sin^2\theta - 1]^{1/2} \qquad \text{... (ii)}$$

During this time, the horizontal distance moved is $2h$. The horizontal velocity being $2\sqrt{(gh)}\cos\theta$.

Using $s = ut$, we have

$$2h = 2\sqrt{(gh)}\cos\theta\cdot t$$

or $$2h = 2\sqrt{(gh)}\cos\theta \times \sqrt{(8h/g}\,[2\sin^2\theta - 1]^{1/2}$$

Squaring $1 = 8\cos^2\theta\,(1 - 2\cos^2\theta)$

or $$16\cos^4\theta - 8\cos^2\theta + 1 = 0,$$

or $$(4\cos^2\theta - 1)^2 = 0.$$

$$\therefore \quad \cos^2\theta = \frac{1}{4} \text{ and } \sin^2\theta = \frac{3}{4}. \qquad \text{... (iii)}$$

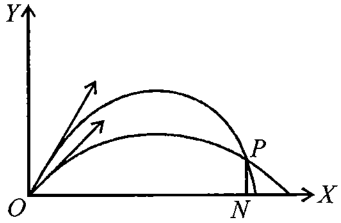
Substituting these values in equation (ii) we have

$$(t_1 - t_2) = \sqrt{[8h/g]}\left[2\cdot\frac{3}{4} - 1\right]^{1/2}$$

or $$t_1 - t_2 = 2\sqrt{(h/g)}.$$

7. Two particles are projected from a point at the same instant with velocities whose horizontal and vertical components are $u_1, v_1$ and $u_2, v_2$ respectively. Prove that the interval between their passing through the other common point of their path is $\frac{2(v_1u_2-v_2u_1)}{g(u_1+u_2)}$.

**Soln.** The situation is shown in figure.



Let the two particles pass through the common point $P$. Again let, $t_1$ and $t_2$ be the time taken by the particles to go to $P$ from $O$.

Along X-axis, we have

$$u_1t_1=u_2t_2 \quad \text{...(i)}$$

In the vertical direction

$$v_1t_1-\frac{1}{2}gt_1^2=PN=v_2t_2-\frac{1}{2}gt_2^2 \quad \text{...(ii)}$$

From equation (ii), we have

$$t_1^2-t_2^2=2/g\cdot(v_1t_1-v_2t_2) \quad \text{...(iii)}$$

But from equation (i), $\frac{u_1}{u_2}=\frac{t_2}{t_1}$.

Addition one to both sides, we have $\frac{u_1+u_2}{u_2}=\frac{t_2+t_1}{t_1}$.

$$\therefore \quad (t_1+t_2)=\frac{(u_1+u_2)}{u_2}\times t_1 \quad \text{...(iv)}$$

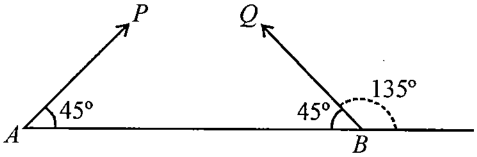
From equation (iii), we get

$$(t_1+t_2)(t_1-t_2)=\frac{2}{g}\cdot(v_1t_1-v_2t_2)(t_1-t_2)\times t_1=\frac{2}{8}\left(v_1t_1-v_2\frac{u_1}{u_2}\cdot t_1\right)$$

$$(t_1+t_2)\frac{(u_1+u_2)}{u_2}=\frac{2}{g}\left(v_1-v_2\frac{u_1}{u_2}\right)(t_1+t_2)=\frac{2}{g}\cdot\frac{u_2}{(u_1+u_2)}\times\frac{(v_1u_2-v_2u_1)}{u_2}$$

$$\text{or} \quad (t_1-t_2)=\frac{2}{g}\frac{(v_1u_2-v_2u_1)}{(u_1+u_2)}.$$

8. Particles $P$ and $Q$ of mass 20 gm and 40 gm respectively are simultaneously projected from points $A$ and $B$ on the ground. The initial velocities of $P$ and $Q$ make 45° and 135° angles respectively with the horizontal as shown in the figure. Each particle has an initial speed of 49 m/sec. The separation $AB$ is 245 m. Both particles travel in the same vertical plane and undergo a collision. After the collision $P$ retraces its path. Determine the position of $Q$ when it hits the ground. How much time after the collision does the particle $Q$ take to reach the ground. Take $g=9.8$ m/sec$^2$.



**Soln.** The horizontal velocity of either particle is $49\cos 45° = 49/\sqrt{2}$ m/sec. Initially, the vertical component of velocity of either particle $= 49\sin 45° = 49/\sqrt{2}$ m/sec. The horizontal distance travelled by each particle at the time of collision = 245/2 metre. The vertical component of the velocity of each particle at the time of collision is

$$\frac{49}{\sqrt{2}} - 9.8\times\frac{245}{2}\times\frac{\sqrt{2}}{49} = 0.$$

At the time of collision, the velocity of each particle is horizontal and is equal to $49/\sqrt{2}$ m/sec.

Let $v_p$ and $v_q$ be the velocities after collision, then

$$0.02\times\frac{49}{\sqrt{2}} - 0.04\times\frac{49}{\sqrt{2}} = 0.02\,v_p + 0.04\,v_q.$$

As $P$ retraces its path, $v_p = -\frac{49}{\sqrt{2}}$.

$$\therefore\quad \frac{49}{\sqrt{2}}(0.02-0.04) = -0.02\times\frac{49}{\sqrt{2}} + 0.04\,v_q$$

The particle falls vertically downward from the point of collision, i.e., at a distance of 122.5 metre from $A$.

Let $t$ be the time taken, then $\frac{49}{\sqrt{2}} = 9.8\times t$.

Since collision does not after the vertical component of velocities.

$$\therefore\quad t = \frac{49}{\sqrt{2}\times 9.8} = \frac{5}{\sqrt{2}} = 3.53 \text{ sec.}$$

9. Two shots are projected from a gun at the top of a hill with the same velocity $u$ at angles of projection $\alpha$ and $\beta$ respectively. If the shots strike the horizontal ground through the foot of the hill at the same point, show that the height $h$ of the hill above the plane is given by

$$h = \frac{2u^2(1-\tan\alpha\tan\beta)}{g(\tan\alpha+\tan\beta)^2}.$$

**Soln.** Let $R$ be the horizontal range for both. Taking the point of projection as origin, the point struck is $(R, h)$. This point satisfies the equations of both the trajectories.

$$-h = R\tan\alpha - \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} \qquad \text{...(i)}$$

and $$-h = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta} \qquad \text{...(ii)}$$

From these equations, we have

$$R\tan\alpha - \frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha} = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

or $$R(\tan\alpha-\tan\beta)=\frac{1}{2}g\frac{R^2}{u^2\cos^2\alpha}-\frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$2u^2/g\,(\tan\alpha-\tan\beta)=R(\sec^2\alpha-\sec^2\beta)$$

or $$2u^2/g\,(\tan\alpha-\tan\beta)=R[(1+\tan^2\alpha)-(1+\tan^2\beta)]$$

or $$2u^2/g=R(\tan\alpha+\tan\beta). \quad \text{... (iii)}$$

$$R=\frac{2u^2}{g(\tan\alpha+\tan\beta)}$$

Putting the value of $R$ in equation (i), we have

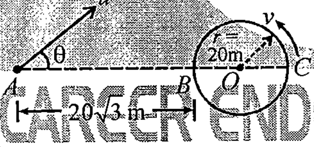
$$h=R\left[\frac{gR}{2u^2\cos^2\alpha}-\tan\alpha\right]=\frac{2u^2}{g(\tan\alpha+\tan\beta)}\left[\frac{\sec^2\alpha}{\tan\alpha+\tan\beta}-\tan\alpha\right]$$

$$=\frac{2u^2}{g(\tan\alpha+\tan\beta)^2}\left[\sec^2\alpha-\tan^2\alpha-\tan\alpha\tan\beta\right]=\frac{2u^2}{g(\tan\alpha+\tan\beta)^2}(1-\tan\alpha\tan\beta).$$

10. A particle is moving along a vertical circle of radius $r=20$ m with a constant speed $v=31.4$ m/s as shown in figure below. Straight line $ABC$ is horizontal and passes through the center of the circle. A shell is fired from point $A$ at the instant when the particle is at $C$. If distance $AB$ is $20\sqrt{3}$ m and the shell collide with the particle at $B$, then prove

$$\tan\theta=\frac{(2n-1)^2}{\sqrt{3}},$$

where $n$ is an integer. Further show that smallest value of $\theta$ is 30°.



**Soln.** As at the time of firing of the shell, the particle was at $C$ and the shell collides with it at $B$, therefore the number of revolutions completed by the particle is odd multiple of half i.e., $(2n-1)/2$, where $n$ is an integer.

Let $T$ be the time period of the particle, then

$$T=\frac{2\pi r}{v}=\frac{2\times3.14\times20}{31.4}=4 \text{ second}$$

If $t$ be the time of flight of the shell, then $t$ = time of $[(2n-1)/2]$ revolutions of the particle

$$=\frac{(2n-1)}{2}\times4=2(2n-1) \text{ second}$$

for a projectile, the time of flight is given by $t=\dfrac{2u\sin\theta}{g}$.

Hence, $$\frac{2u\sin\theta}{g}=2(2n-1) \quad \text{... (i)}$$

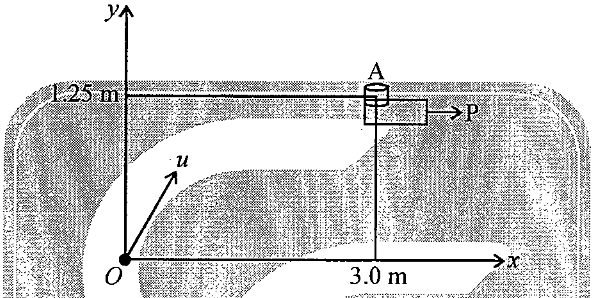
The range of the projectile is given by $R=\dfrac{u^2\sin2\theta}{g}$.

Hence, $\dfrac{u^2 \sin 2\theta}{g} = 20\sqrt{3}$ ... (ii)

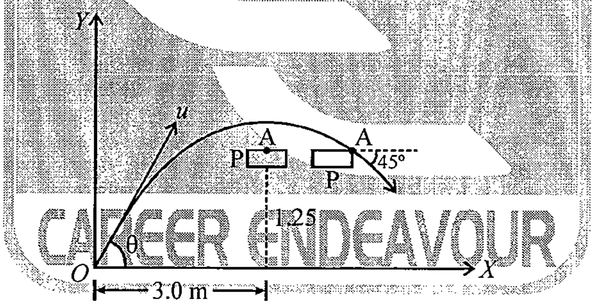
From equations (i) and (ii), we get $\tan\theta = \dfrac{(2n-1)^2}{\sqrt{3}}$.

For θ to be smallest, $n = 1$, so $\theta_{\min} = 30°$.

11. An object $A$ is kept fixed at the point $x = 3\,\text{m}$ and $y = 1.25\,\text{m}$ on a plank $P$ raised above the ground. At time $t = 0$ the plank starts moving along the $+x$ direction with an acceleration 1.5 m/s². At the same instant a stone is projected from the origin with a velocity $u$ as shown. A stationary person on the ground observes the stone hitting the object during its downward motion at an angle of 45° to the horizontal. All the motions are in the $xy$-plane. Find $u$ and the time after which the stone hits the object. Take $g = 10\ \text{m/s}^2$.



**Soln.** See figure.



Let θ be the angle of projectile. Further, suppose the stone hits the object at time $t$, then

$$(u\cos\theta)t = 3.0 + \frac{1}{2}\times 1.5\times t^2 \quad \text{... (i)}$$

and $$(u\sin\theta)t - \frac{1}{2}gt^2 = 1.25 \quad \text{... (ii)}$$

As the stone hits the object $A$, moving along 45° with the horizontal during downward motion, we have

$$\tan(-45°) = \left(\frac{v_y}{v_x}\right) = \frac{u\sin\theta - gt}{u\cos\theta}$$

or $$\frac{u\sin\theta - gt}{u\cos\theta} = -1$$

or $$u\sin\theta - gt = -u\cos\theta \quad \text{... (iii)}$$

From equation (i), $u \cos\theta = \frac{3.0}{t} + 0.75t$

From equation (ii), $u \sin\theta = \frac{1.25}{t} + \frac{g}{2}t$

Substituting these values in equation (iii), we get

$$\frac{1.25}{t} + 5t - 10t = -\frac{3.0}{t} - 0.75t$$

$$\frac{4.25}{t} = 4.25t \text{ or } t^2 = 1 \text{ or } t = 1 \text{ sec.}$$

From equation (i) and (ii), we get

$$u \cos\theta = 3.0 + 0.75 = 3.75 \qquad \text{... (iv)}$$

and $u \sin\theta = 6.25$ ... (v)

Dividing equation (v) equation (iv), we get

$$\tan\theta = \frac{6.25}{3.75} = \frac{5}{3} \text{ or } \theta = \tan^{-1}\left(\frac{5}{3}\right)$$

Squaring and adding equation (iv) and (v), we get

$$u^2 = (3.75)^2 + (6.25)^2 = (14.06) + (39.06)$$

$$u^2 = 53.12 \text{ or } u = 7.29 \text{ m/s.}$$

12. A rotation disc (figure below) moves in the positive direction of the $x$-axis. Find the $y(x)$ describing the positive of the instantaneous axis of rotation, if at the initial moment the axis $C$ of the disc was located at the point $O$ after which it moved

(a) with a constant velocity $v$, while the disc started rotating counter clockwise with a constant angular acceleration $\alpha$ (the initial angular velocity is equal to zero);

(b) with a constant acceleration $a$ (and the zero initial velocity), the disc rotates counter clockwise with a constant angular velocity $\omega$.

**Soln.** (a) When the disc rotates with constant angular velocity $v$ and counter clockwise angular acceleration $\alpha$ :

We have the relation $\omega = \omega_0 + \alpha t = \alpha t, t = \frac{x}{v}$.

Now consider any point $P$ of the disc at a distance $y$ from the $x$-axis; i.e. $\mathbf{r} = -y\,\mathbf{j}$. An axis passing through $P$ and parallel to the axis of rotation. Now the linear velocity of point $P$.

$$\mathbf{v}_p = \vec{\omega} \times \mathbf{r} \text{ and } \vec{\omega} = \alpha\frac{x}{v}\mathbf{k} = \left(\frac{\alpha x}{v}\right)\mathbf{k} \times y(-\mathbf{j}) = \frac{\alpha x}{v} y\,\mathbf{i};$$

that is, the point $P$ has a velocity along the $x$-axis which is the same as the velocity of any point of the disc $= v$.

Hence, $y = \frac{v^2}{\alpha x}$: (equation of a hyperbola)

(b) When the disc rotates with constant angular velocity $\omega$ and constant acceleration $a$ : In this case

$$x = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\left(\frac{2x}{a}\right)}$$

Hence, $v = at = \sqrt{2xa}$

Again $\mathbf{v}_p = \vec{\omega} \times \mathbf{r} = \vec{\omega}\,\mathbf{k} \times y(-\mathbf{j})$,

Therefore, $|\mathbf{v}| = \omega y$ or $\sqrt{2xa} = \omega y$ or $y = \dfrac{\sqrt{2xa}}{\omega}$, a parabola].

13. A balloon rises from rest on the ground with constant acceleration $g/8$. A stone is dropped when the balloon has risen to a height $H$ meter. Show that the time taken by the stone to reach the ground is given by $2\sqrt{(H/g)}$.

**Soln.** The velocity $v$ of the balloon when it has risen to a height $H$ is given by $v^2 = 0 + 2 \times \dfrac{g}{8} \times H$ taking upward direction positive.

$$\therefore \qquad v = \sqrt{\left\{\frac{(gH)}{4}\right\}} = \frac{\sqrt{(gH)}}{2} \text{ m/sec.}$$

This will also be the velocity of the stone in upward direction when it is dropped.

Taking upward direction position for the stone, $-H = \dfrac{\sqrt{(gH)}}{2}t - \dfrac{1}{2}t - \dfrac{1}{2}gt^2$ $(\because\ g = -g$ and $h = -H)$

or $\qquad gt^2 - \sqrt{(gH)}\,t - 2H = 0$

$$t = \frac{\sqrt{(gH)} \pm \sqrt{\{(gH) + (8gH)\}}}{2g} = \frac{\sqrt{(gH)} \pm 3\sqrt{(gH)}}{2g} = 2\sqrt{\left(\frac{H}{8}\right)} \text{ taking positive value.}$$

14. A particle is projected vertically upwards. Prove that it will be at 3/4 of its greatest height at time which are in the ratio 1 : 3.

**Soln.** We know that the greatest height attained $= u^2/2g$ (where $u$ is the initial velocity). Let $t$ be the time when the particle is at a height $\dfrac{3}{4}\left(\dfrac{u^2}{2g}\right)$.

Using the formula $s = ut + \dfrac{1}{2}gt^2$, we have $\dfrac{3}{4}\left(\dfrac{u^2}{2g}\right) = ut - \dfrac{1}{2}gt^2 \Rightarrow t^2 - \dfrac{2u}{g}t + \dfrac{6u^2}{8g^2} = 0$.

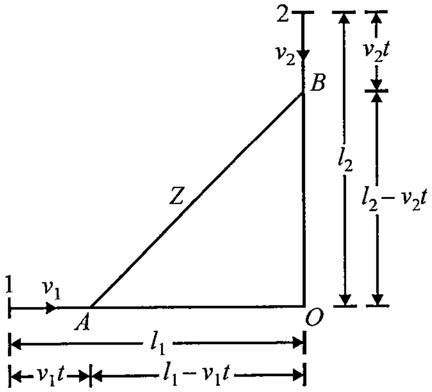
Solving for $t$, we have $t = \dfrac{\dfrac{2u}{g} \pm \sqrt{\left(\dfrac{4u^2}{g^2} - \dfrac{3u^2}{g^2}\right)}}{2} = \dfrac{u}{g} \pm \dfrac{u}{2g}$.

Taking negative sign, $t_1 = \dfrac{u}{2g}$.

Taking only positive sign, $t_2 = \dfrac{3u}{2g}$.

$$\therefore \qquad \frac{t_1}{t_2} = \frac{(u/2g)}{(3u/2g)} = \mathbf{1:3.}$$

15. Two particles, 1 and 2, move with constant velocities $v_1$ and $v_2$ along two mutually perpendicular straight lines towards the intersection point $O$. At the moment $t = 0$ the particles were located at the distances $l_1$ and $l_2$ from the point $O$. How soon will the distance the particles become the smallest ? What is it equal to ?



**Soln.** At a time $t$, the positions of the particle 1 and 2 are separately, $(l_1 - v_1 t, 0)$ and $(0, l_2 - v_2 t)$. Hence the distance between them at this instant.

$$Z = \sqrt{(l_1 - v_1t)^2 + (l_2 - v_2t)^2}$$

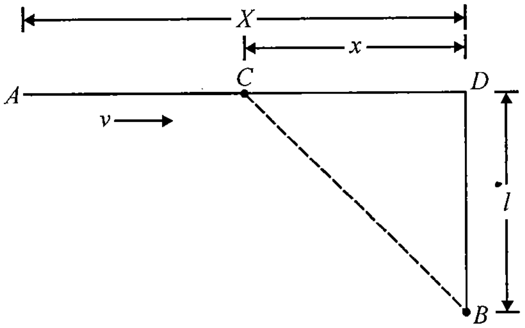
To be minimum at some instant $t = t'$, we use the condition

$$\frac{dZ}{dt} = 0 \Rightarrow \frac{1}{2}\frac{2(l_1 - v_1t)(-v_1) + 2(l_2 - v_2t)(-v_2)}{\sqrt{(l_1 - v_1t)^2 + (l_2 - v_2t)^2}} = 0$$

or $\quad (v_1^2 + v_2^2)t = l_1 v_1 + l_2 v_2$ or $t = t' = \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$

Hence $Z_{\min} = \sqrt{\left(l_1 - v_1 \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)^2 + \left(l_2 - v_2 \dfrac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}\right)^2} = \dfrac{|l_1 v_2 - l_2 v_1|}{\sqrt{(v_1^2 + v_2^2)}}$

16. From point $A$ located on a highway, one has to get by car as soon as possible to point $B$ located in the field at a distance $l$ from the highway. It is known that the car moves in the field $\eta$ times slower than on the highway. At what distance from the point $D$ one must turn off the highway ?



**Soln.** Let the car turn off the road at a distance $x$ from $D$. Hence the total time of travel by car from $A$ to $B$.

$$T = \frac{X - x}{v} + \frac{\sqrt{l^2 + x^2}}{v/\eta}$$

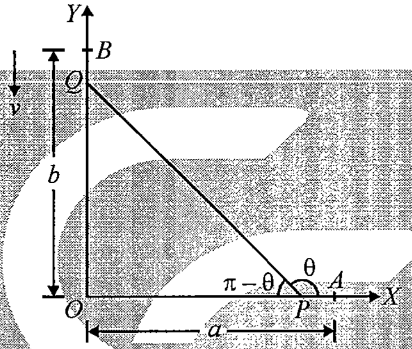
For this to be minimum :

$$\frac{dT}{dx}=0 \Rightarrow -\frac{1}{v}+\frac{\eta}{v}\frac{1}{2}\left(\frac{2x}{\sqrt{l^2+x^2}}\right)=0 \Rightarrow \frac{\eta x}{\sqrt{l^2+x^2}}=1$$

or $$x=\frac{\sqrt{l^2+x^2}}{\eta} \Rightarrow x^2\left(1-\frac{1}{\eta^2}\right)=\frac{l^2}{\eta^2} \Rightarrow x=\frac{l}{\sqrt{\eta^2-1}}.$$

17. Two points are moving with uniform velocities $u$ and $v$ along the perpendicular axes, $OX$ and $OY$. The motion is directed towards $O$, the origin. When $t=0$, they are at a distance $a$ and $b$ respectively from $O$. Calculate the angular velocity of the line joining them at time $t$. Show that it is greatest, when $t=\frac{au+bv}{u^2+v^2}$.

**Soln.** See figure. At $t=0$, the particles are at $A$ and $B$ respectively.



Let at time $t$, the particles are at $P$ and $Q$ respectively.

From figure, $OP=a-ut$ ... (i)

and $OQ=b-vt$ ... (ii)

Let $\theta$ be the angle which the line $PQ$ makes with the direction $OX$ at time $t$. From $\Delta QOP$,

$$\tan(\pi-\theta)=\frac{OQ}{OP}=\frac{b-vt}{a-ut} \text{ or } -\tan\theta=\frac{b-vt}{a-ut}$$

or $$\tan\theta=\left(\frac{b-vt}{ut-a}\right) \text{ or } \theta=\tan^{-1}\left(\frac{b-vt}{ut-a}\right) \quad \text{... (iii)}$$

Differentiating equation (iii), we get $\frac{d\theta}{dt}=\frac{1}{1+\left(\frac{b-vt}{ut-a}\right)^2}\times\frac{(ut-a)(-v)-(b-vt)u}{(ut-a)^2}$

Simiplifying it, we get $\omega=\frac{d\theta}{dt}=\frac{av-bu}{(a-ut)^2+(b-vt)^2}$, $\omega$ will be maximum when denominator is minimum because numerator is constant.

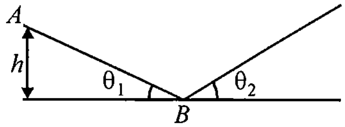
Let $Z=(a-ut)^2+(b-vt)^2$

$$\frac{dZ}{dt}=2(a-ut)(-u)+2(b-vt)(-v)=0$$

Solving it for $t$, we get $t=\frac{au+bv}{u^2+v^2}$.

It can be seen that $\frac{d^2Z}{dt^2}$ is positive. Hence $\omega$ is maximum, when $t=\frac{au+bv}{u^2+v^2}$.

18. A particle of mass $m$ starts from rest moving down the inclined plane $AB$ and rises up to point $C$ on the inclined plane $BC$. Assuming no losses of energy, the time period of the to and for motion of the particle is............



**Soln.** The time taken $t_1$ in moving from $A$ to $B$, is given by $\frac{h}{\sin\theta_1}=0+\frac{1}{2}(g\sin\theta_1)\;\;(\because s=AB=\frac{h}{\sin\theta_1})$.

Therefore, $t_1=\sqrt{\left(\frac{2h}{g}\right)\cdot\frac{1}{\sin\theta_1}}$

Similarly, the time taken, $t_2$ from $C$ to $B$ is given by $t_2=\sqrt{\left(\frac{2h}{g}\right)\cdot\frac{1}{\sin\theta_2}}$.

Time period, $T=2(t_1+t_2)=2\sqrt{\left(\frac{2h}{g}\right)}\left[\frac{1}{\sin\theta_1}+\frac{1}{\sin\theta_2}\right]$.

19. A particle moves in the plane $xy$ with constant acceleration $a$ directed along the negative $y$-axis. The equation of motion of the particle has the form $y=px-qx^2$ where $p$ and $q$ are positive constants. Find the velocity of the particle at the origin of coordinates.

**Soln.** Given that $y=px-qx^2$

$$\therefore\quad \frac{dy}{dt}=p\frac{dx}{dt}-q\cdot 2x\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2}=p\frac{d^2x}{dt^2}-2qx\frac{d^2x}{dt^2}-2q\left(\frac{dx}{dt}\right)^2 \text{ or } -a=-2q\left(\frac{dx}{dt}\right)^2=-2qv_x^2$$

$$\because\quad \frac{d^2x}{dt^2}=0 \text{ (no acceleration along } x\text{-axis), and } \frac{d^2y}{dt^2}=-a.$$

$$\therefore\quad v_x^2=\frac{a}{2q} \text{ or } v_x=\sqrt{\frac{a}{2q}}$$

Further, $\left(\frac{dy}{dt}\right)_{x=0}=p\frac{dx}{dt}$ or $v_y=pv_x$

$$\therefore\quad v_y=p\sqrt{\left(\frac{a}{2q}\right)}$$

Now, $v=\sqrt{(v_x^2+v_y^2)}=\sqrt{\left(\frac{a}{2q}+\frac{ap^2}{2q}\right)}$ or $v=\sqrt{\left[\frac{a(p^2+1)}{2q}\right]}$.

# Chapter 2

## Newton's Laws of Motion

" *What we know is a drop, what we don't know is an ocean.*" **Isaac Newton**

**First Law:** It states that a particle which is not acted upon by any external force, maintains its state of rest or state of uniform motion.

When Newton's law is applied to an object or system which consists of more than one particles, it is centre of mass of the object or system which obeys Newton's law. For example if two particles are placed on a smooth table at some separation, because of their natural gravitational attraction the two will move towards each other but centre of the system of two particles remains at rest.

**Second Law:** It states that rate of change of momentum of a system is equal to resultant external force on the system.

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \dots (2.1)$$

This form of Newton's law is valid in all cases.

If $m$ be mass of system and $\vec{v}$ be its velocity then $\vec{p} = m\vec{v}$

Therefore, $\vec{F} = \frac{d}{dt}(m\vec{v})$

If mass of system is constant then, $\vec{F} = m\frac{d\vec{v}}{dt}$

or $\qquad \vec{F} = m\vec{a} \qquad \dots (2.2)$

where $\vec{a} = \frac{d\vec{v}}{dt}$ is acceleration, evidently, equation (2.2) can be used only if mass is constant. If mass is variable, we use equation (2.1).

If $F_x, F_y, F_z$ be Cartesian components of $\vec{F}$ and $(x, y, z)$ be coordinate of the particle on which force is acting then equation (2.2) can be written in component form as

$$\left.\begin{aligned} F_x &= m\frac{d^2x}{dt^2} \\ F_y &= m\frac{d^2y}{dt^2} \\ F_z &= m\frac{d^2z}{dt^2} \end{aligned}\right\} \qquad \dots (2.3)$$

However, instead of Cartesian coordinates if we use some curvilinear coordinate then we do not get similar relations. For example if $F_r, F_\theta, F_\phi$ be components of force $\vec{F}$ in spherical polar coordinate and $r, \theta, \phi$ be spherical polar coordinates of particle then,

$$\left.\begin{aligned} F_r &\neq m\frac{d^2r}{dt^2} \\ F_\theta &\neq m\frac{d^2\theta}{dt^2} \\ F_\phi &\neq m\frac{d^2\phi}{dt^2} \end{aligned}\right\} \qquad \ldots (2.4)$$

Thus, we cannot write Newton's law in other coordinate system by merely replacing $x, y, z$ by corresponding coordinates.
Due to this reason we say that Newton's law is not covariant under coordinate transformation.

**Third Law:** It states that to every action there is equal and opposite reaction. If $\vec{F}_{12}$ be force on an object 1 due to an object 2, and $F_{21}$ be force on 2 due to 1 then

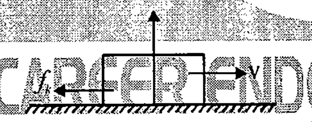
$$\vec{F}_{12} = -\vec{F}_{21}$$

Newton's third law assumes that to an action there is prompt reaction. This implies that in Newton's law it is assumed that information travels with infinite speed.

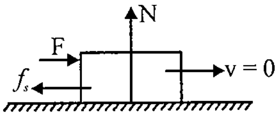
**Violation of Newton's Laws:** Newton's 1st and 2nd laws are strictly valid in inertial frames. In non-inertial frames Newton's laws do not hold good. In non-inertial frames we can apply Newton's law if we include pseudo forces.
There are some cases where Newton's third law also does not hold. For example magnetic force exerted by two non-parallel current carrying wires on each other are not in opposite in direction.

**Friction force:** Its arise because of interaction (electromagnetic) among the particles of surfaces in contact. It is of two types

**(i) Kinetic friction:** It acts on objects which are sliding on a surface $f_k = \mu_k N$



**(ii) Static friction:** It acts on object which are subjected to a force but not sliding on the contact surface. It is an adjustable force $f_s \leq \mu_s N$



**Example :** A block slides down a rough inclined plane of inclination $\theta$. Coefficient of friction between the block and the plane is $\mu$. Calculate acceleration of block.

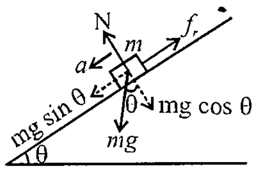
**Soln.** The block slides on the inclined plane.
Therefore, to get acceleration of block we resolve all forces parallel and perpendicular to the plane.
Therefore, for equilibrium perpendicular to the plane

$$N = mg\cos\theta \qquad \ldots \text{(i)}$$

equation of motion parallel to plane, $mg\sin\theta - f_r = ma$ ... (ii)

and, $f_r = \mu N = \mu\, mg\cos\theta$

from (ii), $mg\sin\theta - \mu mg\cos\theta = ma \quad \therefore\ a = g(\sin\theta - \mu\cos\theta)$



**Example:** A block of mass '*m*' is pulled with a force F at an angle $\theta$ with horizontal. The block does not move on the surface. Calculate force of friction on the block and value of coefficient of friction.

**Soln.** The block is stationary, therefore horizontal components of F must be balanced by friction force.

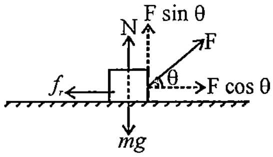
$\therefore f_r = F\cos\theta$

for vertical equilibrium, $N + F\sin\theta = mg$

or, $N = mg - F\sin\theta$

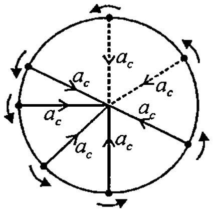
if $\mu$ be coefficient of static friction then,

$f_r \le \mu N$



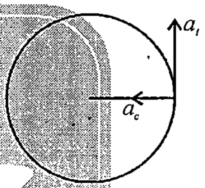
or $F\cos\theta \le \mu(mg - F\sin\theta)$ $\quad \therefore \mu \ge \dfrac{F\cos\theta}{mg - F\sin\theta}$

**Centripetal acceleration:** When a particle moves in a circular path, due to continuous change in direction of velocity of the particle, particle always have an acceleration towards centre of circle.



This is called centripetal acceleration. Its value is $a_c = \dfrac{v^2}{r}$

If the particle's velocity is non-uniform then in addition to centripetal acceleration, the particle also has a tangential acceleration which is equal to $a_t = \dfrac{dv}{dt}$



**Centripetal force:** It is not a new kind of force. When an object moves in a circular path the net components of known forces towards centre of circle is called centripetal force.

It turns out that if a particle of mass *m* is moving in a circle of radius *r*. Then at any instant when its speed is v, net force towards centre $= ma_c = \dfrac{mv^2}{r}$.

**Example :** A point mass *m* is tied to a light string of length '*l*'. It is whirled in a vertical circle. Its speeds at lowest point, highest point and at a point where string makes 60° with upward vertical are $\sqrt{6gl}, \sqrt{2gl}$ and $\sqrt{3gl}$ respectively. Calculate tension in string at these points.

**Soln.** The point mass is moving in a circle, therefore at each point of circle net force towards centre must be equal to $\dfrac{mv^2}{r}$.

**At lowest point:** Net force towards centre $= T - mg$

$$\therefore \; T - mg = \frac{mv^2}{l} \text{ or } T = mg + \frac{m6gl}{l} = 7mg$$

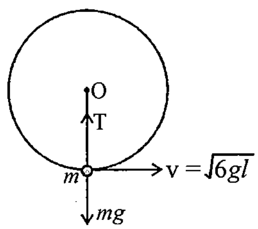
**At highest point:** Net force towards centre $= T + mg$

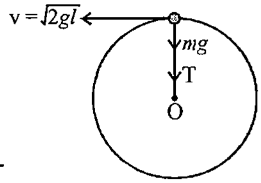
$$\therefore T + mg = \frac{mv^2}{l} \text{ or } T = \frac{m2gl}{l} - mg = mg$$

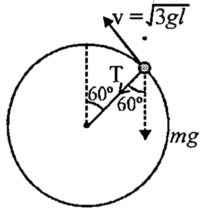
**At 60° with upward vertical:** Net force towards centre $= T + mg\cos 60°$

$$\therefore T + mg\cos 60° = \frac{mv^2}{l}$$

$$T = \frac{m.3gl}{l} - \frac{mg}{2} = \frac{5mg}{2}$$

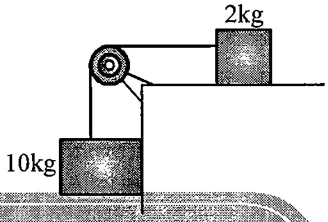

O
T
m
v = $\sqrt{6gl}$
mg


v = $\sqrt{2gl}$
mg
T
O


v = $\sqrt{3gl}$
T
60°
60°
mg

## SOLVED PROBLEMS

1. In the figure, the tension in the inelastic string is T when all the surfaces are frictionless. If 2kg block is glued to the surface, the tension in the string will be **[GP-2007]**



(a) Zero (b) Greater than T

(c) Less than T (d) Equal to T

**Soln.** (b)

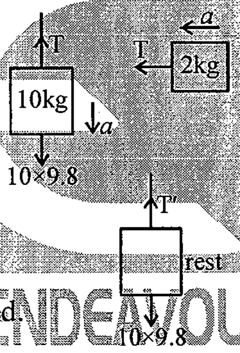
When all surfaces are smooth the two blocks will accelerate. If '$a$' be the acceleration

then, $10 \times 9.8 - T = 10a$

and $T = 2a$

$$\therefore \quad \frac{98-T}{T} = \frac{5}{1}$$

$$T = \frac{98}{6} = 16.33N$$


T
10kg
a
10×9.8
a
T
2kg
T'
rest
10×9.8

when 2kg is glued,

$$T' = 10 \times 9.8 = 98\text{N}$$

$\therefore$ Tension in string increases when 2kg block is glued.

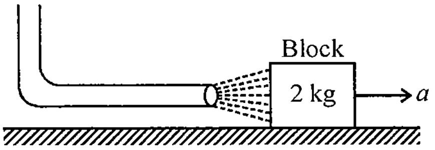
2. A block of metal weighting 2 kg is resting on a frictionless plane. It is struck by a jet releasing water at at rate of 1 kg/s and at a speed of 5 m/s. Calculate the initial acceleration of the block.

**Soln.** $F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$

$$\therefore \quad F = v\frac{dm}{dt} \qquad (\because v = \text{constant})$$

$$= 5 \times 1 = 5 \text{ N}$$

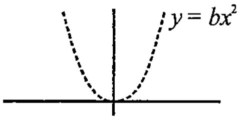
$$F = ma \text{ or } a = \frac{F}{m} = \frac{5}{2} = 2.5 \text{ m/s}^2.$$



3. A particle of mass moves along a curve on a horizontal surface. The equation of curve is $y = bx^2$. Without using curvature formula find radius of curvature at $x = 0$.

**Soln.** Radial acceleration of particle at $x = 0$ is

$$a_y = \left.\frac{d^2y}{dt^2}\right|_{x=0}$$



$$= \left[\frac{d^2}{dt^2}(bx^2)\right]_{x=0} = \left[2b(\dot{x}^2 + x\ddot{x})\right]_{x=0} = 2b\dot{x}^2$$

Since at $x = 0$, $\dot{x} = v_0$

$$\therefore \quad a_y = 2bv_0^2$$

Since radial acceleration is given as

$$a_y = \frac{v_0^2}{r}$$

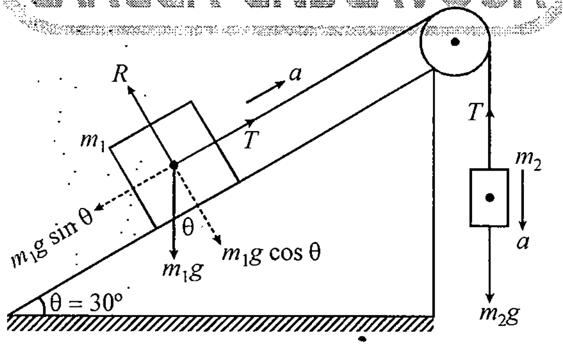
where '$r$' is radius of curvature

$$\therefore \quad 2bv_0^2 = \frac{v_0^2}{r}$$

$$\therefore \quad r = \frac{1}{2b}$$

4. A block of mass $m_1 = 4$ kg on a smooth inclined plane of 30° is connected by a cord over a small, frictionless pulley to a second block of mass $m_2 = 5$ kg hanging vertically. Calculate the acceleration with which the block moves and also the tension in the cord. Take $g = 10$ m/sec².

**Soln.** The different forces acting on the masses are shown in figure.



We have $T - m_1g\sin\theta = m_1a$ ... (i)

and $m_2g - T = m_2a$ ... (ii)

Solving equations (i) and (ii), we get

$$a = \frac{(m_2 - m_1\sin\theta)g}{(m_1 + m_2)} \quad \text{... (iii)}$$

and $$T = m_2 g\left[1 - \frac{(m_2 - m_1 \sin\theta)}{(m_1 + m_2)}\right]$$

or $$T = \frac{m_1 m_2 (1 + \sin\theta) g}{(m_1 + m_2)} \quad \text{... (iv)}$$

Here $m_1 = 4\,\text{kg},\ m_2 = 5\,\text{kg},\ \theta = 30°$ and $g = 10\,\text{m/s}$

Substituting these values in equation (iii), we get

$$T = \frac{5 \times 4\left(1 + \frac{1}{2}\right)10}{9} = \frac{300}{9} = 33.33\ \text{N}.$$

5. A uniform rope of length $L$ is pulled by a constant force $F$. What is the tension in the rope at a distance $l$ from the end, where it is applied ?

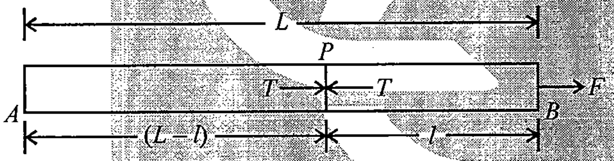
**Soln.** See figure below,

For part $PB$,

$$F - T = (\text{Mass of } PB)\,a, \text{ where } a = \text{acceleration.}$$

or $$F - T = \left(\frac{M}{L}\right) \times l \times a, \quad \text{... (i)}$$

where $\frac{M}{L}$ is mass per unit length.



For part $AP$,

$$T = (\text{Mass of } AP)\,a = \left(\frac{M}{L}\right)(L - l)\,a. \quad \text{... (ii)}$$

Solving the two, we get $a = T\frac{L}{M}(L - l)$.

Substituting $a$ in equation (i) and simplifying, we get

$$T = \left(1 - \frac{l}{L}\right)F$$

6. At the moment $t = 0$ a stationary particle of mass $m$ experiences a time-dependent force $F = kt(t' - t)$, where $k$ is a constant vector, $t'$ is the time during which the given force acts. Find :

(a) the momentum of the particle when the action of the force discontinued ;

(b) the distance covered by particle while the force acted.

**Soln.** (a) Since $F = kt(t' - t)$, the direction of the force is fixed, as $k$ is a constant vector and the particle starts from rest, the motion will be in the direction of the force.

$$m\frac{dv}{dt} = kt(t' - t), \text{ from Newton's II law}$$

Integrating, $\int_0^v dv = \frac{k}{m}\int_0^{t'} t\,(t'-t)\,dt = \frac{k}{m}\left[t'\frac{t^2}{2}-\frac{t^3}{3}\right]_0^{t'}$ $\qquad v = \frac{k}{m}\cdot\frac{t'^3}{6}$

Therefore, momentum of the particle $p = mv = kt\frac{t'^3}{6}$.

(b) $ds = v\,dt \Rightarrow s = \int_0^s ds = \int_0^{t'} v\,dt = \frac{kt'^4}{12m}$.

7. A motor-boat of mass $m$ moves along a lake with velocity $v_0$. At the moment $t = 0$ the engine of the boat is shut down. Assuming the resistance of the particle to be proportional to the velocity of the boat $F = -rv$, find:

(a) how long the motorboat moved with the shutdown engine;

(b) the velocity of the motorboat as a function of the distance covered with the shutdown engine, as well as total distance covered till the complete stop.

(c) the mean velocity of the motorboat over the time interval (beginning with the moment $t = 0$), during which its velocity decreases $\eta$ times.

**Soln.** (a) $F = rv \Rightarrow m\frac{dv}{dt} = -rv \Rightarrow \frac{dv}{v} = -\frac{r}{m}dt$

Integrating, $\log_e v = -\frac{r}{m}t + C$

At $t = 0,\ v = v_0,\ \therefore C = \log_e v_0$

Therefore, $v = v_0\,e^{-rt/m}$ ; $v \to 0$ as $t \to \infty$

Hence, the velocity becomes zero after in infinite time.

(b) From above, $m\frac{dv}{dt} = m\frac{dv}{ds}\cdot\frac{ds}{dt} = mv\frac{dv}{ds} = -rv$ or $dv = -\frac{r}{m}ds$

Integrating, we get $v = -\frac{r}{m}s + C_1$

But $v = v_0$ when $s = 0,\ \therefore C_1 = v_0$

Therefore, $v = v_0 - \frac{rs}{m}$

Total distance covered : $s_0 = \frac{mv_0}{r}$, by putting $v = 0$ at $s = s_0$

(c) Let the velocity becomes $\frac{v_0}{\eta}$ at time $t'$,

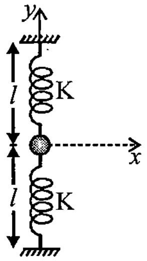
$\therefore \quad \frac{v_0}{\eta} = v_0\,e^{-rt'/m} \Rightarrow t' = -\frac{m}{r}\log_e\frac{1}{\eta} = \frac{m}{r}\log_e\eta$

$\therefore \quad \bar{v}$ = average velocity during time interval $t'$

$$= \frac{\int_0^{t'} v(t)\,dt}{t'} = \frac{-\frac{m}{r}v_0\,e^{-rt/m}}{t'} \text{ or } \bar{v} = \frac{-\frac{m}{r}v_0\left(e^{-rt'/m}-1\right)}{\frac{m}{r}\log_e\eta} = \left.\frac{v_0(\eta-1)}{\eta\log_e\eta}\right].$$

8. A particle of mass $m$ is connected to two springs of unstretched length '$l$' and spring constant $k$ as shown in figure. Calculate acceleration of particle if it is slightly displaced along X-direction.

**Soln.** Let the particle is displaced by a distance $x$ along $+x$ direction as shown in figure.

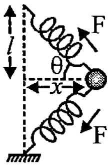


Therefore, stretched length of spring $= \sqrt{l^2+x^2}$

elongation in springs $= \sqrt{l^2+x^2}-l$

Restoring force on the particle due to each spring $F = k(\sqrt{l^2+x^2}-l)$

Because of restoring force the particle moves back towards its initial position.



Equation of motion of particle $F_x = m\dfrac{d^2x}{dt^2}$

or $-2F\cos\theta = m\dfrac{d^2x}{dt^2}$ or $-2k(\sqrt{l^2+x^2}-l).\dfrac{x}{\sqrt{l^2+x^2}} = m\dfrac{d^2x}{dt^2}$

Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left(1-\dfrac{l}{\sqrt{l^2+x^2}}\right)$ or $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1-\left(1+\dfrac{x^2}{l^2}\right)^{-1/2}\right]$

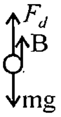
Since, $x<<l, \left(1+\dfrac{x^2}{l^2}\right)^{-1/2} \approx 1-\dfrac{x^2}{2l^2}$

Therefore, $\dfrac{d^2x}{dt^2} = -\dfrac{2kx}{m}\left[1-\left(1-\dfrac{x^2}{2l^2}\right)\right]$ or $\dfrac{d^2x}{dt^2} = -\dfrac{kx^3}{ml^2}$

9. Spherical particles of a given material of density $\rho$ are released from rest inside a liquid medium of lower density. The viscous drag force may be approximated by the Stoke's law, i.e, $F_d = 6\pi\eta R\mathrm{v}$, where $\eta$ is the viscosity of the medium, R the radius of a particle and v its instantaneous velocity. If $\tau(m)$ is the time taken by a particle of mass $m$ to reach half its terminal velocity, then the ratio $\tau(8m)/\tau(m)$ is

**Soln.** Each particle has same density but different radii and masses. We have to calculate time of fall in terms of mass of particle therefore we will write our equations explicitly in terms of mass and we will remove radius from our equations wherever it appears.

Drag force on particles



$F_d = 6\pi\eta R\mathrm{v}$, Mass of a particle $m = \rho.\dfrac{4}{3}\pi R^3$

$\therefore R = \left(\dfrac{3m}{4\pi\rho}\right)^{1/3}$ $\therefore F_d = 6\pi\eta\left(\dfrac{3m}{4\pi\rho}\right)^{1/3} v = Km^{1/3}v$, where $K = 6\pi\eta\left(\dfrac{3}{4\pi\rho}\right)^{1/3}$

If B is buoyancy force, then equation of motion of a particle is

$m\dfrac{dv}{dt} = mg - F_d - B$ where B $= \dfrac{4}{3}\pi R^3\sigma g = \dfrac{4}{3}\pi R^3\rho g.\dfrac{\sigma}{\rho} = mg\dfrac{\sigma}{\rho}$, $\sigma$ = density of medium

$$\therefore m\frac{dv}{dt} = mg - Km^{1/3}v - mg\frac{\sigma}{\rho}, \quad m\frac{dv}{dt} = mg\left(1-\frac{\sigma}{\rho}\right) - Km^{1/3}v$$

or $\frac{dv}{dt} = g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v$ ... (a)

when terminal velocity is reached $\frac{dv}{dt} = 0$

$$\therefore 0 = g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v_t \qquad \therefore v_t = \frac{g\left(1-\frac{\sigma}{\rho}\right)}{Km^{-2/3}} \qquad \text{...(b)}$$

Now, if $\tau$ be the time to reach half the terminal velocity then from (a)

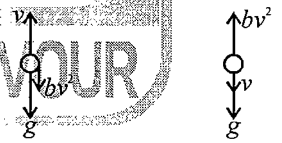
$$\int_0^{v_t/2} \frac{dv}{g\left(1-\frac{\sigma}{\rho}\right) - Km^{-2/3}v} = \int_0^{\tau} dt, \quad \therefore -\frac{1}{Km^{-2/3}}\ln\left[\frac{g\left(1-\frac{\sigma}{\rho}\right) - \frac{Km^{-2/3}v_t}{2}}{g\left(1-\frac{\sigma}{\rho}\right)}\right] = \tau$$

using value of $v_t$ from (b) we get, $\tau = \frac{m^{2/3}}{K}\ln 2$ or $\tau(m) = \frac{m^{2/3}\ln 2}{K}$

$$\therefore \tau(8m)/\tau(m) = (8m)^{2/3}/m^{2/3} = 4$$

10. A particle of unit mass is thrown vertically upward with initial speed $v_0$. It is acted upon by a drag force $bv^2$ in addition to gravity where $b$ is constant and $v$ is instantaneous velocity of particle. Calculate speed of the particle when it returns to the point from where it was thrown.

**Soln.** In presence of drag force only quantity that is common in upward and downward motion is the distance covered.



upward motion downward motion

For upward motion, initial speed = $v_0$,

final speed = 0, let height reached = $h$

equation of motion $\frac{dv}{dt} = -g - bv^2$

or $\frac{dv}{dy}.\frac{dy}{dt} = -g - bv^3$, $\frac{dy}{dt} = v$ or $\frac{vdv}{g+bv^2} = -dy$

$$\therefore \int_{v_0}^{0} \frac{vdv}{g+bv^2} = -\int_0^h dy$$

on integration we get, $\frac{1}{2b}\ln\left(\frac{g+bv_0^2}{g}\right) = h$ ... (a)

for downward motion, initial speed = 0, final speed = $v$(let), height descended = $h$

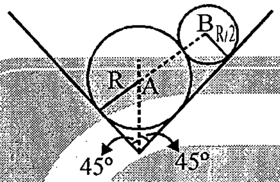
equation of motion $\frac{dv}{dt} = g - bv^2$ or $\frac{vdv}{g-bv^2} = dy$ or $\int_0^v \frac{vdv}{g-bv^2} = \int_0^h dy$

on integration we get, $\frac{1}{2b}\ln\left(\frac{g}{g-bv^2}\right)=h$ ... (b)

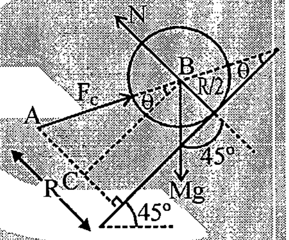
Comparing (a) and (b) we get, $\frac{g+bv_0^2}{g}=\frac{g}{g-bv^2}$, $g-bv^2=\frac{g^2}{g+bv_0^2}$ or $v=\frac{v_0}{\sqrt{1+\frac{bv_0^2}{g}}}\Rightarrow v<v_0$

due to drag force the particle returns with a speed less than its initial speed. If drag force were absent $b=0$, then $v=v_0$. Therefore in absence of drag force particle returns with same speed as its initial value.

11. Two solid spheres each of mass M and radii R and $\frac{R}{2}$ stay in equilibrium between two perpendicular smooth planks as shown in the figure. Find the value of contact force between the two spheres.



**Soln.** Line of their contact force will pass through their centres.
Free body diagrams of smaller sphere is shown in figure.
In triangle ACB



$AC=R/2$

$AB=R+\frac{R}{2}=\frac{3R}{2}$

$\therefore\ CB=R\sqrt{2}$

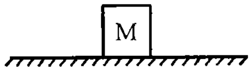
$\therefore\ \cos\theta=\frac{CB}{AB}=\frac{2\sqrt{2}}{3}$

Component of forces parallel to plank surfaces give

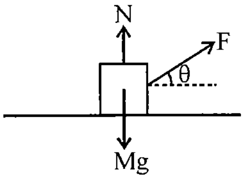
$F_c\cos\theta-Mg\sin 45°=0$

$\therefore\ F_c=\frac{Mg/\sqrt{2}}{\cos\theta}=\frac{3Mg}{4}$

12. A block of mass M lies on a rough surface. If $\mu_s$ be coefficient of static friction between the block and the surface. Calculate the minimum force required to move the block.



**Soln.** Let the force is applied at an angle $\theta$ with the horizontal



$\therefore$ For minimum value of F which can make the block move.

$N+F\sin\theta=Mg$

or $N=Mg-F\sin\theta$ ... (i)

$F\cos\theta=\mu N$

or $F\cos\theta = \mu(Mg - F\sin\theta)$

$$\therefore \quad F = \frac{\mu Mg}{\cos\theta + \mu\sin\theta} \qquad \text{... (ii)}$$

For F to be minimum $(\cos\theta + \mu\sin\theta)$ should be maximum

$$\therefore \quad \frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = 0$$

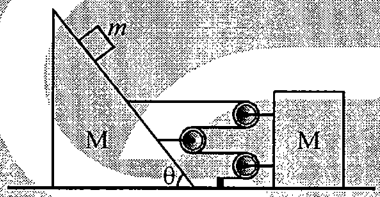
$$-\sin\theta + \mu\sin\theta = 0$$

$$\therefore \quad \tan\theta = u$$

$$\therefore \quad \sin\theta = \frac{\mu}{\sqrt{\mu^2+1}} \text{ and } \cos\theta = \frac{1}{\sqrt{1+\mu^2}}$$

$\therefore$ from (ii) we get $F = \dfrac{\mu Mg}{\sqrt{1+\mu^2}}$

13. In the figure shown, find the acceleration of triangular block. All the surfaces are smooth.



**Soln.** As block *m* slides triangular block will move towards left due to which rectangular block will also move towards left.

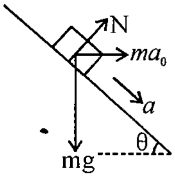
Let $a_0$ be acceleration of triangular block.

When triangular block moves, length of 3 segments of string changes whereas when rectangular block moves, length of four segments changes. Therefore, acceleration of rectangular block $= \frac{3}{4}a_0$.

Let T be tension in string.

$\therefore$ For rectangular block $\quad 4T = M\left(\frac{3}{4}a_0\right)$

$$\therefore \quad T = \frac{3Ma_0}{16}$$



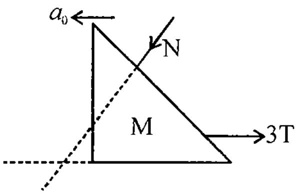
For *m*: (free body diagram is shown in figure)

$N = mg\cos\theta - ma_0\sin\theta$

For triangular block

$N\sin\theta - 3T = Ma_0$

$$m(g\cos\theta - a_0\sin\theta)\sin\theta - \frac{9Ma_0}{16} = Ma_0$$



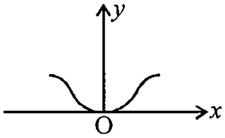
$$mg\sin\theta\cdot\cos\theta=\left(\frac{25}{16}M+m\sin^2\theta\right)a_0 \qquad \therefore \quad a_0=\frac{g\sin\theta\cdot\cos\theta}{\left(\sin^2\theta+\frac{25M}{16m}\right)}$$

14. A bead of mass $m$ moves with constant speed $v_0$ along a wire having shape $y=b\cos\frac{2\pi x}{L}$ kept vertically. Calculate normal reaction due to the wire when bead is at $x=0$.

**Soln.** Equation of motion of bead at $x = 0$

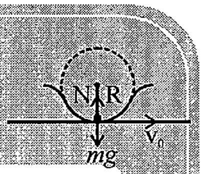
$$N-mg=\frac{mv_0^2}{R}$$

where N is normal reaction and R is radius of curvature.



$$\therefore \quad N=\frac{mv_0^2}{R}+mg \qquad \text{... (i)}$$

Here, $\frac{dx}{dt}=v_0$ and $\ddot{x}=0$ at $x=0$



and $\frac{d^2y}{dt^2}=-\frac{v_0^2}{R}$ at $x=0$

$$\therefore \quad \frac{d}{dt}\left(-\dot{x}\frac{2\pi b}{L}\sin\frac{2\pi}{L}x\right)_{x=0}=-\frac{v_0^2}{R}$$

$$\text{or}\left[-\ddot{x}\frac{2\pi b}{L}\sin\frac{2\pi}{L}x-\dot{x}^2\left(\frac{2\pi}{L}\right)^2 b\cos\frac{2\pi}{L}x\right]_{x=0}=\frac{v_0^2}{R}$$

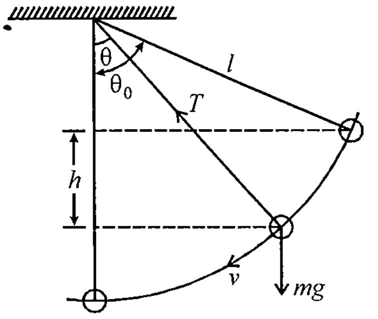
$$\Rightarrow \quad R=\frac{L^2}{4\pi^2 b}$$

$\therefore$ from (i) we get

$$N=\frac{mv_0^2}{L^2}\cdot 4\pi^2 b+mg \quad =m\left[\left(\frac{2\pi}{L}v_0\right)^2 b+g\right]$$

15. A 40 kg mass, hanging at the end of a rope of length $l$, oscillates in a vertical plane with an angular amplitude of $\theta_0$. What is the tension in the rope, when it makes an angle $\theta$ with the vertical ? If the breaking strength of the rope is 80 kg f, what is the maximum angular amplitude with which the mass can oscillate without the rope breaking ?

**Soln.** The situation is shown in figure below.



(a) From figure $h = l(\cos\theta - \cos\theta_0)$

and $v^2 = 2gh = 2gl(\cos\theta - \cos\theta_0)$ ... (i)

Again $T - mg\cos\theta = mv^2/l$ ... (ii)

Substituting the value of $v^2$ from equation (i) in equation (ii), we get

$$T - mg\cos\theta = m\{2gl(\cos\theta - \cos\theta_0)/l\}$$

or $T = mg\cos\theta + 2mg(\cos\theta - \cos\theta_0)$

or $T = mg(3\cos\theta - 2\cos\theta_0)$

or $T = 40g(3\cos\theta - 2\cos\theta_0)$ newton

or $T = 40(3\cos\theta - 2\cos\theta_0)$ kgf

(b) Let $\theta_0$ be the maximum amplitude. The maximum tension $T$ will be at mean position where $\theta = 0$.

$\therefore$ $T_{\max} = 40(3 - 2\cos\theta_0)$

But $T_{\max} = 80$

Solving we get $\theta_0 = 60°$.

16. A smooth table is placed horizontally and an ideal spring of spring constant $k = 1000$ N/m and unextended length of 0.5 m has one end fixed to its center. The other end is attached to a mass of 5 kg which is moving in a circle with constant speed 10 m/s. Find the tension in the spring and the extension of the spring beyond its normal length.

**Soln.** Let $T$ be the tension produced in the streched string. The centripetal force required for the mass $m$ to move in a circle is provided by the tension $T$. The streched length of the spring is $r$ (= radius of the circle). Now,

Elongation produced in the spring $= (r - l_0)$

Tension produced in the spring $T = k(r - l_0)$ ... (i)

where $k$ is spring constant.

Centripetal force $= \dfrac{mv}{r}$

Therefore, $k(r - l_0) = \dfrac{mv^2}{r}$ or $kr(r - l_0) = mv^2$ ... (ii)

Substituting the given values, we get

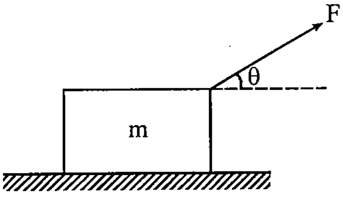
$$\begin{aligned} 1000\,r(r - 0.5) &= 5 \times 100 \\ r(r - 0.5) &= 0.5 \\ \text{or} \quad 2r^2 - r - 1 &= 0 \end{aligned}$$

Solving, we get $t = 1.0$ m

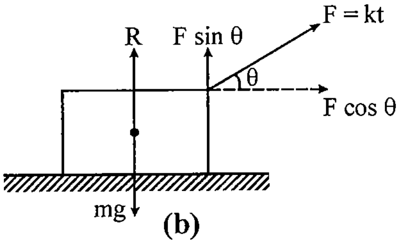
Now, the extension of the spring beyond its normal length is $r - l_0 = 1.0 - 0.5 = 0.5$ m

From equation (i), $T = 1000 \times 0.5 = 500$ N.

17. At the moment $t = 0$, the force $F = kt$ is applied to a small body of mass $m$ resting on smooth horizontal plane ($k$ = constant). The permanent direction of this force forms an angle $\theta$ with the horizontal (figure below). Find



(a)



(b)

(a) the velocity of the body at the moment of its breaking off the plane;
(b) the distance traversed by the body up to this moment.

**Soln.** The free body diagram is shown in figure.

From figure : $R + F\sin\theta = mg$

At breaking off, $R = 0$, $\therefore F = \dfrac{mg}{\sin\theta}$

Now $kt = \dfrac{mg}{\sin\theta}$ or $t = \dfrac{mg}{k\sin\theta}$

(a) If $a$ be the acceleration, then $F\cos\theta = ma$

Therefore, $kt\cos\theta = m \times \left(\dfrac{dv}{dt}\right)$ $\left(\because a = \dfrac{dv}{dt}\right)$.

Integrating this expression within proper limits, we get $m\int_0^v dv = k\cos\theta \int_0^{mg/k\sin\theta} t\,dt$

or $mv = \dfrac{k\cos\theta}{2}\left[\dfrac{mg}{k\sin\theta}\right]^2$ or $v = \dfrac{mg^2}{2k}\left(\dfrac{\cos\theta}{\sin^2\theta}\right)$

(b) Without putting the limits, we have $v = \dfrac{k\cos\theta}{2m}t^2 + C$, where $C$ = constant of integration.

When $t = 0$, $v = 0$ and hence $C = 0$.

Now, $\dfrac{ds}{dt} = \dfrac{k\cos\theta}{2m}t^2$ $\int_0^s ds = \dfrac{k\cos\theta}{2m}\int_0^{mg/k\sin\theta} t^2\,dt$

$s = \dfrac{k\cos\theta}{2m}\left[\dfrac{t^3}{3}\right]_0^{mg/k\sin\theta}$ or $s = \dfrac{m^2g^3\cos\theta}{6k^2\sin^3\theta}$.

18. A bar of mass $m$ resting on a smooth horizontal plane starts moving due to a force $F = mg/3$ of constant magnitude. In the process of its rectilinear motion, the angle $\theta$ between the direction of this force and the horizontal varies as $\theta = ks$, where $k$ is a constant. Find the velocity of the bar as a function of angle $\theta$.

**Soln.** The situation is shown in figure below.

From free body diagram

$$F\sin\theta + R = mg \qquad \text{... (i)}$$

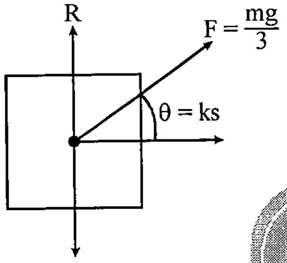
and $F\cos\theta = ma$ ... (ii)

From equation (ii),

$$\left(\frac{mg}{3}\right)\cos ks = m\left(\frac{dv}{dt}\right) = mv\left(\frac{dv}{ds}\right)$$

or $$\int_0^v v\,dv = \frac{g}{3}\int \cos ks\,ds$$

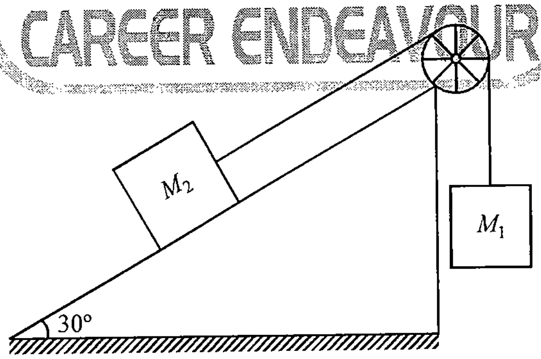
or $$\frac{v^2}{2} = \frac{g}{3k}\sin ks + C$$



But when $v = 0, s = 0 \;\therefore C = 0$

Therefore, $v^2 = \dfrac{2g}{3k}\sin ks$ or $v = \sqrt{\left[\left(\dfrac{2g}{3a}\right)\sin\theta\right]}$.

19. (a) Two masses $M_1$ and $M_2$ are connected by light string, which passes over the top of a smooth plane inclined at 30° to the horizontal, so that one mass rests on the plane and the other hangs vertically as shown in figure. It is found that $M_1$, hanging vertically can draw $M_2$ up the full length of the plane in half the time in which $M_2$ hanging vertically draws $M_1$ up. Find $M_1/M_2$. Assume pulley to be smooth.



(b) Initially at time $t = 0$ smooth masses ($M_1 = 15$ kg and $M_2 = 10$ kg) are held at rest and then they released. If after one second, the string snaps, find the further time taken for the 15 kg mass to return to its original position on the plane.

(Take $g = 10$ m/s$^2$)

**Soln.** (a) Let $T$ be the testing in the string. Then

$$M_1 g - T = M_1 a_1 \quad \text{... (i)}$$

and $$T - M_2 g\sin 30^\circ = M_2 a_1 \quad \text{... (ii)}$$

Solving equations (i) and (ii), we get

$$a_1 = \frac{g(M_1 - M_2 \sin 30^\circ)}{M_1 + M_2} \qquad \text{... (iii)}$$

Further, $M_2 g - T = M_2 a_2$ ... (iv)

and $T - M_1 g \sin 30^\circ = M_1 a_2$ ... (v)

Solving equation (iv) and (v), we get

$$a_2 = \frac{g(M_2 - M_1 \sin 30^\circ)}{M_1 + M_2} \qquad \text{... (vi)}$$

If $s$ be the distance travelled, then $s = \frac{1}{2} a_1 t_1^2 = \frac{1}{2} a_2 t_2^2$.

$$\therefore \quad \frac{a_1}{a_2} = \frac{t_2^2}{t_1^2}$$

$$\text{or} \quad \frac{a_1}{a_2} = 4 \quad \left( \because \frac{t_1}{t_2} = \frac{1}{2} \text{ given} \right) \qquad \text{... (vii)}$$

Dividing equations (iii) and (vi), we have

$$\frac{a_1}{a_2} = \frac{M_1 - M_2 \sin 30^\circ}{M_2 - M_1 \sin 30^\circ} \qquad \text{... (viii)}$$

Substituting the value of $\left( \frac{a_1}{a_2} \right)$ from equation (vii) in equation (viii), we get

$$4 = \frac{M_1 - M_2 \sin 30}{M_2 - M_1 \sin 30} = \frac{M_1 - (M_2/2)}{M_2 - (M_1/2)}$$

$$4 = \frac{2M_1 - M_2}{2M_2 - M_1}$$

$$\text{or} \quad 8M_2 - 4M_1 = 2M_1 - M_2 \quad \text{or} \quad 9M_2 = 6M_1 \quad \text{or} \quad \left( \frac{M_1}{M_2} \right) = \frac{3}{2}.$$

(b) The acceleration $a$ before snapping of the string is given by

$$a = \frac{g(M_2 - M_1 \sin \theta)}{(M_1 + M_2)} = \frac{10(10 - 15/2)}{25} = 1 \text{ m/s}^2$$

Let $s$ be the distance travelled in one second after the release of masses, then

$$s = 0 + \frac{1}{2} a t^2 = \frac{1}{2} \times 1 \times (1)^2 = 0.5 \text{ meter up the plane.}$$

The velocity $v$ will be $v = at = 1 \times 1 = 1$ m/s.

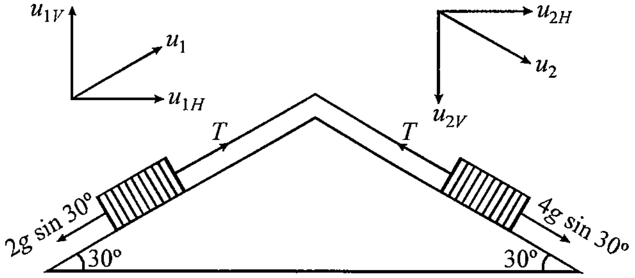
After snapping of the string, let $t$ be the time taken by the mass 15 kg to return to its original position. The distance travelled would be $-0.5$ m. Its initial velocity $u$ is 1 m/s and its acceleration $a$ will be $-g \sin 30$.

Now, $s = ut + \frac{1}{2} a t^2$ or $-0.5 = 1 \times t - \frac{1}{2}\left( 10 \times \frac{1}{2} \right) t^2$ or $5t^2 - 2t - 1 = 0$.

Solving we get $t = 0.69$ second.

20. A double inclined plane is in the form of a wedge with fixed horizontal base and smooth faces inclined at angle of 30° to the horizontal. A 4 gm particle is on one face and is connected by a light flexible, inextensible string over the smooth rigid of the wedge to a 2 gm particle on other face. If the particles start moving from the position with the string taut, find the direction and magnitude of the velocity of 2 gm particle relative to the 4 gm one second later, if the string is sufficiently long for the particles still to be on each face on the wedge.

**Soln.** The situation is shown in figure.



Let $a$ be the acceleration produced. The equation of motion of two particles are written as

$$(4\times10^{-3})g\sin 30° - T = (4\times10^{-3})a \quad \text{...(i)}$$

$$T-(2\times10^{-3})g\sin 30° = (2\times10^{-3})a \quad \text{...(ii)}$$

Adding these two equations we get

$$2\times10^{-3}g\sin 30 = (6\times10^{-3})a$$

$$\therefore \quad a=\frac{g}{6}=\frac{9.8}{6}=1.63\ \text{m/s}^2$$

After one second, the speed of particle along the slopes will be 1.63 m/s. Resolving velocities horizontally and vertically, we have

For $2g$ mass,

$$u_{1H}=u_1\cos 30°=1.63\times\left(\frac{\sqrt{3}}{2}\right)\text{m/s}$$

$$u_{1V}=u_1\sin 30°=\left(\frac{1.63}{2}\right)\text{m/s}$$

For $4g$ mass,

$$u_{2H}=u_2\cos 30°=1.63\times\left(\frac{\sqrt{3}}{2}\right)\text{m/s}$$

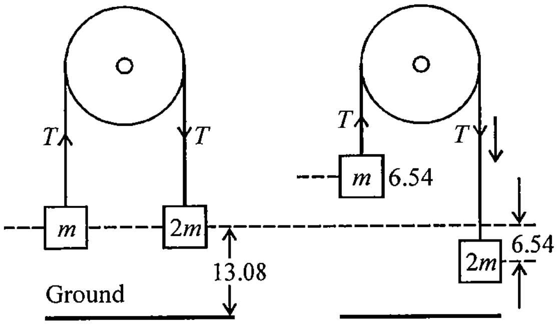
$$u_{2V}=u_2\sin 30°=\left(\frac{1.63}{2}\right)\text{m/s}$$

The horizontal components are equal and in the same direction while the vertical components are equal in magnitude but opposite in direction.

The relative vertical velocity is given by

$$\frac{1.63}{2}-\left(-\frac{1.63}{2}\right)=0.815+0.815=1.63\ \text{m/s upward.}$$

21. Two masses $m$ and $3m$ are connected by a massless string which passes over a light frictionless pulley shown in figure. The masses are initially held with equal lengths of the strings on either side of the pulley. Find the velocity of the masses at the instant the lighter mass moves up a distance of 6.54 meter. This string is suddenly cut at that instant. Calculate the time taken by each mass to reach the ground. Take $g = 981$ cm/sec$^2$.



(a) (b)

**Soln.** Let $T$ be the tension in the string and $a$ be the acceleration of the system. The mass $m$ will move upward and mass $2m$ will go downward with same acceleration. Here we have

$$2mg - T = 2m \times a \qquad \text{...(i)}$$

and $T - mg = ma$ ...(ii)

Solving these equations,

$$a = \frac{g}{3} = \frac{981}{3} = 327 \text{ cm/sec}^2 \text{ or } a = 3.27 \text{ m/sec}^2.$$

For lighter mass, initial velocity $= u$, distance moved $= 6.54$ m, acceleration $a = 3.27$ m/sec$^2$, final velocity $v = ?$

Now $v^2 = u^2 + 2as = 0 + 2 \times 3.27 \times 6.54$ or $v^2 = 6.54 \times 6.54$ $\therefore v = 6.54$ m/sec.

Similarly, the velocity of heavior mass = 6.54 m/sec in downward direction.

When the string is cut, the distance of lighter mass from ground = 13.08 + 6.54 = 19.62 meter and the distance of heavier mass from the ground = 13.08 − 6.54 = 6.54 m. Let $t_1$ and $t_2$ be the time taken by the lighter and heavier masses to reach the ground respectively.

For lighter mass $s_1 = 19.62$ m, $u = -6.54$ m/sec, $g = 9.81$ m/sec$^2$, $t_1 = ?$

Using $s = ut + \frac{1}{2}gt^2$, we have $19.62 = \frac{1}{2} \times 9.8\, t_1^2 - 6.54\, t_1$

or $3t_1^2 + 4t_1 - 4 = 0$ or $(3t_1 - 2)(t_1 + 2) = 0$

As $t$ can not be negative, hence $t_1 = \frac{2}{3}$ seconds.

For heavier mass, $s_2 = 6.54$ m, $u = 6.54$ m/sec.

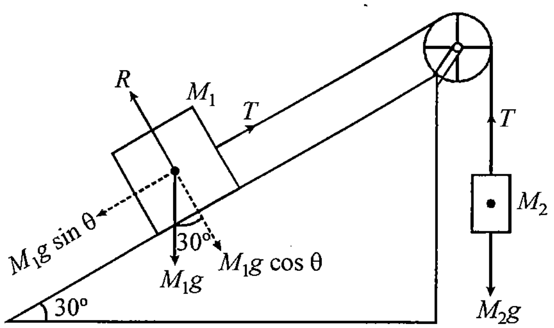
Therefore, $6.54 = 6.54 \times t_2 + \frac{1}{2} \times 9.81\, t_2^2$

$$3t_2^2 + 4t_2 - 4 = 0$$

Solving we get = 2.88 seconds.

22. A wire of $9.8 \times 10^{-3}$ kg mass per meter meter passes over a frictionless pulley fixed on the top of an inclined frictionless plane which makes an angle of 30° with the horizontal. Masses $M_1$ and $M_2$ are tied at the two ends of the wire. The mass $M_1$ rests on the plane and the mass $M_2$ hangs freely vertically downwards. The whole system is in equilibrium. Now a transverse wave propagates along the wire with a velocity of 100 m/sec. Find $M_1$ and $M_2$.

**Soln.** The situation is shown in figure.



Here, we have

$$T = M_1 g \sin 30° \quad \text{...(i)}$$

$$R = M_1 g \cos\theta \quad \text{...(ii)}$$

and $$T = M_2 g \quad \text{...(iii)}$$

From equation (i) and (iii), we have

$$M_1 g \sin 30° = M_2 g \text{ or } \frac{M_1}{2} = M_2 \text{ or } M_1 = 2M_2$$

Velocity of transverse wave $v = \sqrt{\frac{T}{m}}$, where $m$ is the mass per unit length.

$$\therefore \quad v^2 = \frac{T}{m} \text{ or } T = v^2 m$$

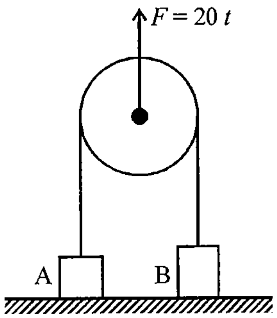
or $$T = (100 \text{ m/s})^2 \times (9.8 \times 10^{-3} \text{ kg/m}) = 98 \text{ newton} \quad \text{...(v)}$$

From equation (iii), $M_2 = \frac{T}{g} = \frac{98 \text{ newton}}{9.8 \text{ newton/kg}} = 10 \text{ kg}$

From equation (iv), $M_1 = 2 \times M_2 = 2 \times 10 \text{ kg} = 20 \text{ kg}$.

23. Two blocks $A$ and $B$ of mass 1 kg and 2 kg respectively are connected by a string, passing over a light frictionless pulley. Both the blocks are resting on a horizontal floor and the pulley is held such that string remains just taut.

At moment $t = 0$, a force $F = 20t$ newton starts acting on the pulley along vertically upward direction as shown in figure. Calculate



(a) Velocity of $A$ and $B$ when loses contact with the floor.
(b) Height raised by the pulley upto that instant and
(c) Workdone by the force $F$ upto that instant $g = 10\ \text{m/s}^2$.

**Soln.** (a) Let $T$ be the tension in the string. Then, $2T = 20t$ or $T = 10t$ newton.

Let the block $A$ loses its contact with the floor at time $t = t_1$. This happens when the tension in string becomes equal to the weight of $A$. Thus

$$T = mg \text{ or } 10t_1 = 1 \times 10 \text{ or } t_1 = 1 \text{ sec.} \quad \text{... (i)}$$

Similarly, for block $B$, we have

$$10t_2 = 2 \times 10 \text{ or } t_2 = 2 \text{ sec} \quad \text{... (ii)}$$

i.e., the block $B$ loses contact after 2 seconds. For block $A$, at time $t$ such that $t_1 \le t$, let $a$ be the its acceleration in upward direction. Then

$$10t - 1 \times 10 = 1 \times a = \left(\frac{dv}{dt}\right)$$

$$\text{or} \quad dv = 10(t-1)\,dt \quad \text{... (iii)}$$

Integrating this expression, we get

$$\int_0^v dv = 10 \int_{t_1}^{t} (t-1)\,dt$$

$$\text{or} \quad v = 5t^2 - 10t + 5 \quad \text{... (iv)}$$

Now velocity $v$ at $t = 2$ sec is $v = 20 - 20 + 5 = 5$ m/s. ... (v)

(b) From equation (i)

$$dy = (5t^2 - 10t + 5)\,dt \quad \text{... (vi)}$$

where $y$ is the vertical displacement of block $A$ at time $t\,(> t_1)$

Integrating equation (vi), we get

$$\int_{y=0}^{y=h} dy = \int_{t=1}^{t=2} (5t^2 - 10t + 5)\,dt$$

$$h = 5\left[\frac{t^3}{3}\right]_1^2 - 10\left[\frac{t^2}{2}\right]_1^2 + 5[t]_1^2 = \frac{5}{3}\text{ m}$$

Therefore, height raised by pulley upto that instant $= \frac{h}{2} = \frac{5}{6}$ m.

(c) The workdone by the force $W =$ increased in potential energy $(m\,g\,h)$ of block $A +$ K.E. of the block

$$= m\,g\,h + \frac{1}{2}mv^2$$

$$= 1 \times 10 \times \left(\frac{5}{3}\right) + \frac{1}{2} \times 1 \times (25) = \left(\frac{175}{6}\right)\text{joule.}$$

24. A block is projected up with 10 m/s along a fixed inclined plane of inclination 37° with the horizontal. If the time of ascend from the point of projection is half the time of descend to the same point, find the distance travelled by the block during the up and down journey.

**Soln.** Let $\mu$, $t_1$ and $t_2$ be the coefficient of friction between the plane and the block, time of ascend and time of descend respectively.

The retardation while going up

$$a_1 = g\,(\sin\theta + \mu\cos\theta) = 10\left(\frac{3}{5} + \mu\frac{4}{5}\right).$$

The acceleration while descending

$$a_2 = g\,(\sin\theta - \mu\cos\theta) = 10\left(\frac{3}{5} - \mu\frac{4}{5}\right).$$

Now, $s =$ distance of ascend $=$ distance of descend.

As final velocity is zero, we have

$$0 = u - a_1 t_1 \text{ or } u = a_1 t_1$$

Now, $s = a_1 t_1 - \frac{1}{2}a_1 t_1^2 = \frac{1}{2}a_1 t_1^2$

Then, $s = \frac{1}{2}a_1 t_1^2 = \frac{1}{2}a_2 t_2^2$ and $t_2 = 2t_1$

$$\therefore \quad \left(\frac{a_2}{a_1}\right) = \left(\frac{t_1}{t_2}\right)^2 = \left(\frac{1}{2}\right)^2 \text{ or } \frac{\left(\frac{3}{5} - \mu\frac{4}{5}\right)}{\left(\frac{3}{5} + \mu\frac{4}{5}\right)} = \left(\frac{1}{2}\right)^2$$

Solving we get $\mu = \left(\frac{9}{20}\right)$

Again $a_1 = 10\left(\frac{3}{5} + \frac{9}{20} \times \frac{4}{5}\right) = 9.6\text{ m/sec}^2$

$$\therefore \quad s = \frac{u^2}{2a_1} = \frac{(10)^2}{2 \times 9.6} = 5.21\text{ meter}$$

So total distance $= 2s = 10.42$ meter.

25. A block of mass 2 kg slides on an inclined plane which makes an angle of 30° with the horizontal. The coefficients of friction between the block and the surface is $\sqrt{(3/2)}$.

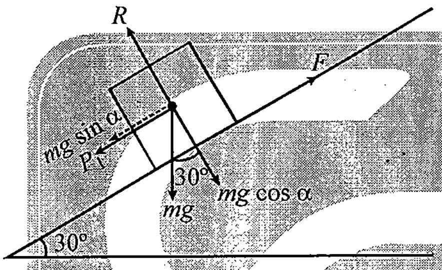
(i) What force should be applied to the block so that the block moves down without any acceleration ?
(ii) What force should be applied to the block so that it moves up without any acceleration ?
(iii) Calculate the ratio of power needed in the above two cases, if the block moves with the same speed in both the cases.

**Soln.** (i) The situation is shown in figure (a), acts upwards along the plane. Let a force $P_1$ be applied in this case. Resolving along and perpendicular to the plane, we have

$$P_1 + mg\sin\alpha = F \qquad \text{... (i)}$$

and $$mg\cos\alpha = R \qquad \text{... (ii)}$$

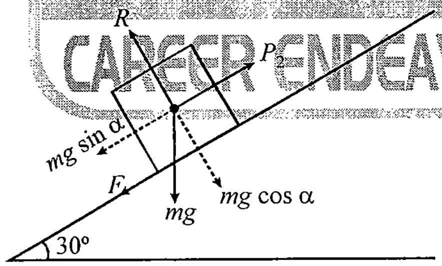
where $F = \mu R = \mu mg\cos\alpha = \frac{\sqrt{3}}{2} \times 2 \times 9.8 \times \frac{\sqrt{3}}{2} = \frac{29.4}{\sqrt{2}}$ N.



(a)

$$\therefore \quad P_1 = F - mg\sin\alpha = \frac{29.4}{\sqrt{2}} - \frac{2\times 9.8\times 1}{2} = \frac{29.4}{\sqrt{2}} - 9.8 = 11 \text{ newton.}$$

(ii) The situation is shown in figure (b).



(b)

Let $P_2$ be the force applied upwards in this case. Resolving along and perpendicular to the plane, we have

$$P_2 = mg\sin\alpha + F$$

$$R = mg\cos\alpha$$

$$\because \quad F = \frac{29.4}{\sqrt{2}} \text{ newton}$$

$$\therefore \quad P_2 = 2\times 9.8\times\frac{1}{2} + \frac{29.4}{\sqrt{2}} = 9.8 + \frac{29.4}{\sqrt{2}} = 30.6 \text{ newton.}$$

(iii) We know that, power = force × velocity.

Since velocity is same in both the cases, the ratio of powers will be equal to the ratio of forces applied.

Therefore, ratio $= \dfrac{P_1}{P_2} = \dfrac{11}{30.6} = 0.36.$

26. A small bar starts sliding down an inclined plane forming an angle $\theta$ with the horizontal. The friction coefficient depends on the distance $x$ covered as $\mu = kx$, where $k$ is a constant. Find the distance covered by the bar till it stops and its maximum velocity over this distance.

**Soln.** The net acceleration of the body

$$a = \frac{mg\sin\theta - \mu\, mg\cos\theta}{m} = g\sin\theta - \mu g\cos\theta$$

$$= g\sin\theta - (kx)g\cos\theta \;\; = g(\sin\theta - kx\cos\theta) \qquad \text{... (i)}$$

We know that

$$a = \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt} = v\times\frac{dv}{dx}$$

$$\therefore \quad v\times\frac{dv}{dx} = g(\sin\theta - kx\cos\theta)$$

$$\text{or} \quad v\,dv = g(\sin\theta - kx\cos\theta)\,dx \qquad \text{... (ii)}$$

Integrating this expression, we get

$$\frac{v^2}{2} = (g\sin\theta)x - gk\frac{x^2}{2}\cos\theta + C \qquad \text{... (iii)}$$

where $C$ is integration constant.

When $x = 0$, $v = 0$ and hence $C = 0$

$$\frac{v^2}{2} = (g\sin\theta)x - gk\left(\frac{x^2}{2}\right)\cos\theta \qquad \text{... (iv)}$$

When $v = 0$

$$x\left[(g\sin\theta) - gk\left(\frac{x}{2}\right)\cos\theta\right] = 0$$

either $x = 0$ or $g\sin\theta - gk\left(\dfrac{x}{2}\right)\cos\theta = 0$

$$\therefore \quad x = \frac{2\sin\theta}{k\cos\theta} = \frac{2}{k}\tan\theta \qquad \text{... (v)}$$

For maximum velocity, $\left(\dfrac{dv}{dx}\right) = 0$

From equation (iv),

$$v\frac{dv}{dx} = g\sin\theta - gkx\cos\theta = 0$$

$$\text{or} \quad x = \frac{\sin\theta}{k\cos\theta} = \frac{1}{k}\tan\theta \qquad \text{... (vi)}$$

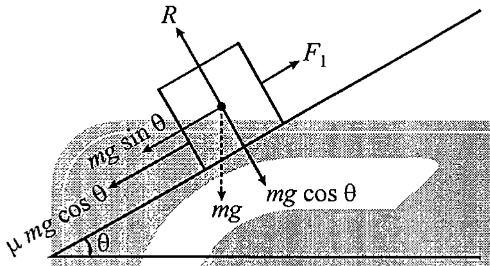
So, at a distance $x = \frac{1}{k}\tan\theta$, the velocity will be maximum. Putting this value in equation (iv), we get

$$\frac{v^2}{2} = (g\sin\theta)\,(\tan\frac{\theta}{k}) - gk\,(\tan^2\frac{\theta}{2k^2})\cos\theta$$

Solving we get $v_{\max} = \sqrt{\left[\left(\frac{g}{k}\right)\sin\theta\tan\theta\right]}$.

27. The force required just to move a body up an inclined plane is double the force required just to prevent the body sliding down it. The coefficient of friction is $\mu$. Prove that the angle of inclination of the plane $\theta = \tan^{-1}(3\mu)$.

**Soln.** Let $F_1$ be the force required to move a body up the inclined plane with inclination $\theta$ (see figure (a)).

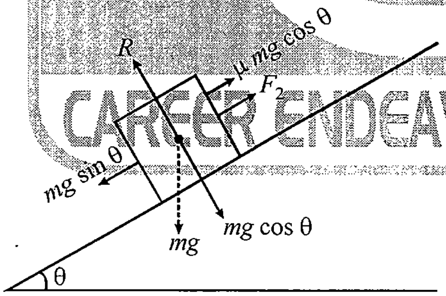


**(a)**

$$F_1 = mg\sin\theta + \mu mg\cos\theta \qquad \text{...(i)}$$

Let $F_2$ be the force required just to prevent the body sliding down. Then as shown in figure (b) we have

$$mg\sin\theta = F_2 + \mu\, mg\cos\theta \qquad \text{...(ii)}$$



**(b)**

According to the problem

$$mg\sin\theta + \mu\, mg\cos\theta = 2[mg\sin\theta - \mu\, mg\cos\theta]$$

or $\sin\theta + \mu\cos\theta = 2\sin\theta - 2\mu\cos\theta$

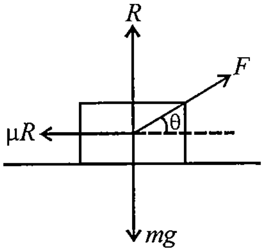
$3\mu\cos\theta = \sin\theta$

$3\mu = \tan\theta$

$\therefore \quad \theta = \tan^{-1}(3\mu)$.

28. A particle of mass $m$ rests on a horizontal floor with it has a coefficient of static friction $\mu$. It is desired to make the body move by applying the minimum possible force $F$. Find the magnitude of $F$ and the direction in which it has to be applied.

**Soln.** The situation in shown in figure. Let $F$ be the force applied at an angle $\theta$ with the horizontal.



The horizontal and vertical components of $F$ are $F\cos\theta$ and $F\sin\theta$ respectively. For horizontal and vertical equilibriums, we have

$$R + F\sin\theta = mg, \qquad \text{... (i)}$$

and $$F\cos\theta = \mu R, \qquad \text{... (ii)}$$

From equation (i), we get $R = (mg - F\sin\theta)$.

Substituting in equation (ii), we get

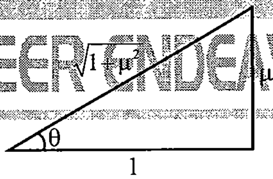
$$F\cos\theta = \mu(mg - F\sin\theta) = \mu mg - \mu F\sin\theta$$

$$F[\cos\theta + \mu\sin\theta] = \mu mg.$$

$$\therefore \quad F = \frac{\mu mg}{\cos\theta + \mu\sin\theta} \qquad \text{... (iii)}$$

For the force to be minimum, the denominator $(\cos\theta + \mu\sin\theta)$ should be maximum.

i.e., $$\frac{d}{d\theta}(\cos\theta + \mu\sin\theta) = 0 - \sin\theta + \mu\cos\theta = 0 \text{ or } \tan\theta = \mu \qquad \text{... (iv)}$$



Now, $\sin\theta = \dfrac{\mu}{\sqrt{(1+\mu^2)}}$ and $\cos\theta = \dfrac{1}{\sqrt{(1+\mu^2)}}$.

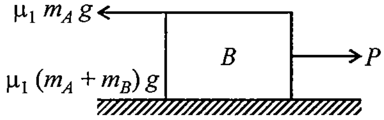
Substituting these values in equation (iii), we get

$$F_{\min} = \frac{\mu mg}{\dfrac{1}{\sqrt{(1+\mu^2)}} + \dfrac{\mu\cdot\mu}{\sqrt{(1+\mu^2)}}} = \frac{\mu mg}{\dfrac{1+\mu^2}{\sqrt{(1+\mu^2)}}} = \frac{\mu mg}{\sqrt{(1+\mu^2)}} \qquad \text{... (v)}$$

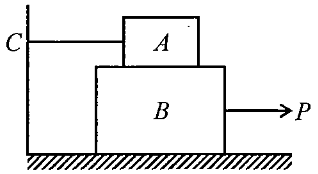
and the angle of direction of force with horizontal.

$$\theta = \tan^{-1}\mu \text{[from equation (iv)]}.$$

29. Block $A$ (mass 200 kg) rests on block $B$ (mass 300 kg) tied with a horizontal string to the wall at $C$ shown in figure (a). The coefficient of friction between $A$ and $B$ is 1/4 and that between $B$ and surface 1/5. What horizontal force $P$ in necessary to move the block $B$.



**(a)**



**(b)**

**Soln.** Frictionless force on $B$ due to $A$,

$$= \mu_1 m_A g = \frac{1}{4} \times 200 \times 10 = 500 \text{ newton.}$$

Frictional force on $B$ due to surface,

$$= \mu_2 (m_A + m_B) g$$

$$= \frac{1}{5}(200 + 300) \times 10 = 1000 \text{ newton.}$$

$$\therefore \quad P = \mu_1 m_A g + \mu_2 (m_A + m_B) g = 500 + 1000 = 1500 \text{ newton.}$$

30. A body of mass $m$ rests on a horizontal plane with the friction coefficient $\mu$. At the moment $t = 0$, a horizontal force is applied to it, which varies with time $F = kt$, where $k$ is a constant vector. Find the distance traversed by the body during first/seconds after the force action began.

**Soln.** Here just after applying the force, the motion does not start due to friction force. As the applied force is proportional to time, let after a time $t_0$, the motion starts.

Now, $F = k t_0 = \mu mg$ or $t_0 = \left(\frac{\mu mg}{k}\right)$

If $t \le t_0$, the distance traversed by the body $s = 0$.

When $t \le t_0$, then $F = k(t - t_0)$

$$\therefore \quad m\frac{dv}{dt} = k(t - t_0) \text{ or } m\,dv = k(t - t_0)\,dt \quad \ldots \text{(i)}$$

Integrating equation (i), we get $mv = \frac{k}{2}(t - t_0)^2 + C_1$

When $t = t_0, v = 0, \therefore C_1 = 0$

$$\therefore \quad mv = \frac{k}{2}(t - t_0)^2 \quad \ldots \text{(ii)}$$

$$\text{Again } m\frac{ds}{dt} = \frac{k}{2}(t - t_0)^2 \text{ or } m\,ds = \frac{k}{2}(t - t_0)^2\,dt \quad \ldots \text{(iii)}$$

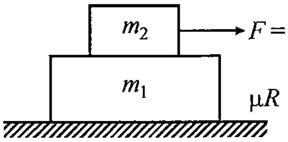
Integrating equation (iii), we get $s = \frac{k}{6m}(t - t_0)^3 + C_2$

Here $C_2 = 0$ because when $t = t_0, s = 0$

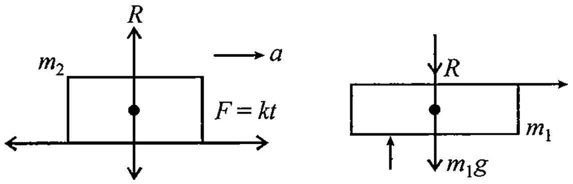
$$\therefore \quad s = \frac{k}{6m}(t - t_0)^3.$$

31. A plank of mass $m_1$ with a bar of mass $m_2$ placed on it lies on a smooth horizontal plane. A horizontal force growing with time $t$ as $F = kt$ ($k$ is constant) is applied to the bar. Find how the acceleration of the plank $a_1$ and of the bar $a_2$ depend on $t$, if the coefficient of friction between the plank and the bar is equal to $\mu$.

**Soln.** The situation is shown in figure (a). Figure (b) shows the free body diagrams of $m_1$ and $m_2$.
It is obvious that if the frictional force is greater than the applied force, then both the blocks will move together. After some time, when the frictional force is less than applied force, then both the blocks will move with different accelerations.



(a)



(b)

(i) Let, both the blocks move together for a time $t_0$. In this case, acceleration $a = \dfrac{F}{m_1 + m_2}$ or $a = \dfrac{k t_0}{m_1 + m_2}$.

From free body diagram of $m_2$, we have

$$F - \mu m_2 g = m_2 a$$

$$k t_0 - \mu m_2 g = \frac{m_2 k t_0}{m_1 + m_2}$$

$$t_0 \left( k - \frac{m_2 k}{m_1 + m_2} \right) = \mu m_2 g$$

$$\therefore \quad t_0 = \frac{\mu m_2 g (m_1 + m_2)}{m_1 k}$$

Hence for $t \le t_0$, both the blocks will move together with acceleration $= \dfrac{kt}{(m_1 + m_2)}$.

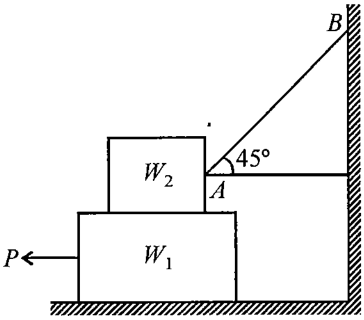
(ii) When applied force on $m_2$ is greater than frictional force :
In this case, both the blocks will move with different accelerations.
Consider the free diagram of body $m_2$. We have $(F - \mu m_2 g) = m_2 a_2$ or $(at - \mu m_2 g) = m_2 a_2$

$$\therefore \quad a_2 = \frac{(at - \mu m_2 g)}{m_2} \text{ where } t \ge t_0.$$

Similarly, for mass $m_1$

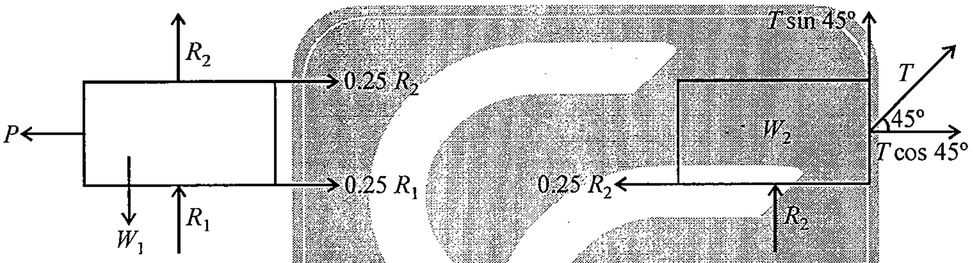
$$a_1 = \frac{(\mu m_2 g)}{m_1} \text{ where } t \ge t_0.$$

32. Calculate the force $P$ required to cause the block of weight $W_1 = 200\text{ N}$ just to slide the block of weight $W_2 = 100\text{ N}$. What is the tension in the string $AB$ ? Coefficient of friction $\mu = 0.25$ for all surfaces in contact.



(a)

**Soln.** The different forces are shown in figure (b).



(b)

For upper mass

$$R_2 + T\sin 45° = 100 \quad \text{...(i)}$$

and $\quad T\cos 45° = 0.25\, R_2 \quad$ ...(ii)

Solving equations (i) and (ii), we get $R_2 = 80$ Newton.

For lower mass, $R_2 + W_1 = R_1$ or $80 + 200 = R_1$

Therefore, $R_1 = 280$ Newton

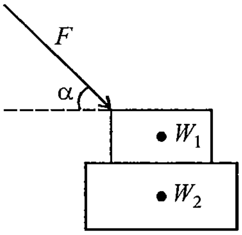
Now, $P = 0.25\, R_1 + 0.25\, R_2 = 0.25\,(R_1 + R_2) = 0.25\,(280 + 80) = 90$ Newton.

From equation (i), $R_2 + T\sin 45° = 100$ or $T\sin 45° = 100 - 80 = 20$

Therefore, $T = \dfrac{20}{\sin 45°} = 20 \times \sqrt{2}$ Newton.

33. In figure, weight $W_1$ is on top of weight $W_2$ which is turn on a horizontal plane. The coefficient of friction between $W_1$ and $W_2$ is $\mu_1$ while that between $W_2$ and the plane is $\mu_2$. Suppose that a force $F$ inclined at an angle $\alpha$ to the horizontal is applied to weight $W_1$. Prove that if $\cot\alpha \geq \mu_1 > \mu_2$, then $W_2$ moves relative to the plane while $W_1$ does not move relative to $W_2$ is that

$$\frac{\mu_2 (W_1 + W_2)}{(\cos\alpha - \mu_2 \sin\alpha)} < F \leq \frac{\mu_1 W_1}{(\cos\alpha - \mu_1 \sin\alpha)}.$$



**Soln.** The following forces are acting on $W_1$:

(i) Weight $W_1$ acting vertically downwards.

(ii) Applied force $F$ at an angle $\alpha$.

(iii) Normal reaction $R$ due to $W_2$.

(iv) Frictional force $F_r$ along the horizontal direction towards left.

Since $W_1$ does not move with respect to $W_2$, hence

$$F_r = F\cos\alpha \quad \text{...(i)}$$

and $$R = W_1 + F\sin\alpha \quad \text{...(ii)}$$

Now $F_r = \mu_1 R = \mu_1(W_1 + F\sin\alpha) = F\cos\alpha$

or $\mu_1 W_1 + \mu_1 F\sin\alpha = F\cos\alpha$

$$\therefore \quad F = \frac{\mu_1 W_1}{\cos\alpha - \mu_1\sin\alpha} \quad \text{...(iii)}$$

If $F$ is less than this, the block $W_1$ would still not move.

Let the block of mass $W_2$ moves horizontally with acceleration $a$. The following forces are acting on this block.

(i) The weight $(W_1 + W_2)$ acting vertically downward.

(ii) Normal reaction $N$ due to floor.

(iii) Applied force $F$ at an angle $\alpha$.

(iv) Frictional force $F_r$

As there is no vertical motion

$$N = F\sin\alpha + W_1 + W_2 \quad \text{...(iv)}$$

As the weight is moving towards the right

$$F\cos\alpha - F_r > 0 \quad \text{...(v)}$$

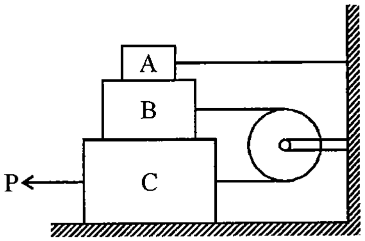
$$F\cos\alpha > F_r > \mu_2(F\sin\alpha + W_1 + W_2)$$

$$F > \frac{(W_1 + W_2)\mu_2}{\cos\alpha - \mu_2\sin\alpha} \quad \text{...(vi)}$$

For $W_2$ to move when $W_1$ is at rest w.r.t. $W_2$

$$\frac{(W_1 + W_2)\mu_2}{(\cos\alpha - \mu_2\sin\alpha)} < F \le \frac{\mu_1 W_1}{(\cos\alpha - \mu_1\sin\alpha)}.$$

34. In figure, the blocks $A$, $B$ and $C$ weight are 3 kg and 8 kg respectively. The coefficient of sliding friction between any two surfaces is 0.25. $A$ is held at rest by a massless rigid rod fixed to the wall, while $B$ and $C$ are connected by a light flexible cord passing around a fixed frictionless pulley. Find the force $P$ necessary to drag $C$ along the horizontal surface to the left at a constant speed. Assume that the arrangement shown in the diagram, $B$ on $C$ and $A$ on $B$ is maintained all throughout.

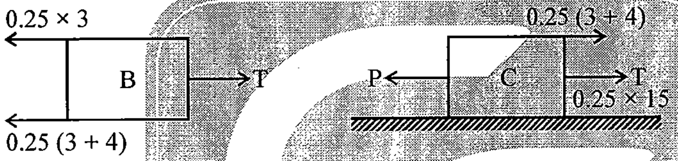


(a)

**Soln.** When block $C$ moves towards left, $B$ moves towards right, while $A$ is fixed. There would be a tension $T$ in the string. Under this condition, let us consider the frictional forces between different surfaces.

Frictional force between $A$ and $B$ $= \mu R = 0.25 \times 3$

Frictional force between $C$ and surface $= 0.25\,(3+4+8) = 0.25 \times 15.$



(b)

Considering figure (b),

Tension in the string = Frictional forces at upper and lower surfaces of block $B$

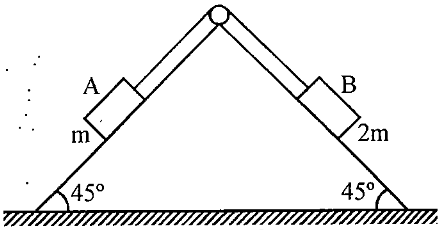
or $T = 0.25 \times 3 + 0.25 \times 7 = 2.5$ kg wt.

35. Block $A$ of mass $m$ and block $B$ and mass $2m$ are placed on a fixed triangular wedge by means of a massless, inextensible string and a frictionless pulley as shown in figure. The wedge is inclined at 45° to the horizontal on both sides. The coefficient of friction between block $A$ and wedge is 2/3 and that between block $B$ and the wedge is 1/3. If the system of $A$ and $B$ is released from rest, find

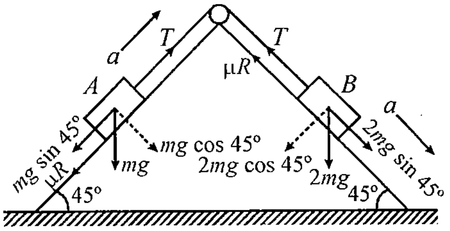
(i) the acceleration of $A$

(ii) tension in the string

(iii) the magnitude and direction of the force of friction acting on $A$.



**Soln.** (i) Let the acceleration of $B$ is in the downward direction. The different forces on the masses are shown in figure.



From figure, for $B$

$$2\,mg\sin 45° - T - \frac{1}{3}\times 2\,mg\,\cos 45° = 2\,ma$$

or $$\sqrt{2}\,mg\sin 45° - T - \frac{\sqrt{2}\,mg}{3} = 2\,ma$$

or $$\frac{2\sqrt{2}\,mg}{3} - T = 2\,ma \qquad \text{... (i)}$$

For $A$, $T - mg\sin 45° - \frac{2}{3}mg\cos 45° = ma$

$$T - \frac{5}{3}\frac{mg}{\sqrt{2}} = ma \qquad \text{... (ii)}$$

Adding equations (i) and (ii), we get $\left(\frac{2\sqrt{2}}{3} - \frac{5}{3\sqrt{2}}\right)mg = 3\,ma$

Solving we get, $a = \frac{g}{9}\left[\frac{4-5}{\sqrt{2}}\right] = -\frac{g}{9\sqrt{2}}$.

Thus, $a$ is negative.

Similarly, if we take the motion of $A$ downward, then too $a$ is negative. This is not possible. Here it should be noted that the body can moved only under the component of weight of the body parallel to the plane but here the resistive force is more than this force. So, acceleration of $A = 0$.

(ii) Now, $T + \frac{1}{3}\times(2\,mg\cos 45) = 2\,mg\sin 45$

$$T = \sqrt{2}\,mg - \frac{\sqrt{2}}{3}mg = \frac{2\sqrt{2}}{3}mg$$

(iii) If we take the frictional force in the upward $T + f = \frac{mg}{\sqrt{2}}$

or $$f = \frac{mg}{\sqrt{2}} - \frac{2\sqrt{2}}{3}mg = -\frac{mg}{3\sqrt{2}}.$$

So, the magnitude of frictional force will be $\frac{mg}{3\sqrt{2}}$ and it will act downward.

36. If the earth be one-half its present distance from the sun, how many days will be in one year ?

**Soln.** Let the radius of the earth's orbit round the sun be $R$. When it is reduced to half, the new distance becomes $R/2$. As the earth is moving round the sun, centripetal force is provided by the gravitational attraction of the sun on the earth.

Hence, $\frac{mv^2}{R} = \frac{GMm}{R^2}$, where the symbols have usual meanings.

$$\therefore \quad v^2 = \frac{Gm}{R} \qquad \text{... (i)}$$

If $T$ be the time period of the earth to complete one revolution round the sun, then

$$v = \frac{2\pi R}{T} \qquad \text{... (ii)}$$

From equations (i) and (ii), we have

$$\frac{4\pi^2 R^2}{T^2} = \frac{GM}{R} \text{ or } T^2 = \frac{4\pi^2 R^3}{GM} \qquad \text{... (iii)}$$

Let $T_1$ be the time period of revolution of earth round the sun when the distance between the two is $R/2$.

$$T_1^2 = \frac{4\pi^2 R^3}{8GM} \qquad \text{... (iv)}$$

Dividing equation (iv) and equation (iii), we have

$$T_1^2 = \frac{T^2}{8} \text{ or } T_1 = \frac{T}{2\sqrt{2}}$$

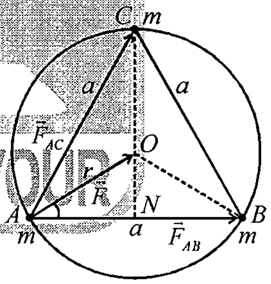
$$\therefore \quad T_1 = \frac{365}{2\sqrt{2}} \quad (\because 1 \text{ year} = 365 \text{ days})$$

or $\quad T_1 = \mathbf{129\ days.}$

37. Three particles, each of mass $m$ are situated at the vertices of an equilateral triangle of side length $a$. The only forces acting on the particles are their mutual gravitational forces. It is desired that each particle moves in a circle while maintaining the original separation $a$. Find the initial velocity that should be given to each particle and also the time period of the circular motion.

**Soln.** The situation is shown in figure. The radius of circular path

$$r = OA = OB = OC = \left(\frac{AN}{\cos 30^\circ}\right) = \frac{a/2}{\sqrt{3}/2} = \frac{a}{\sqrt{3}}.$$



The gravitational force at $A$, due to mass $m$ at $C$

$$F_{AC} = \frac{Gmm}{a^2} \text{along } AC$$

Similarly, the gravitational force at $A$, due to mass $m$ at $B$

$$F_{AB} = \frac{Gmm}{a^2} \text{along } AB$$

The resultant $F$ of these forces would be along $AO$. As the angle between the two forces is 60°, hence the resultant force $F$ is given by

$$F = \sqrt{\left(F_{AC}^2 + F_{AB}^2 + 2F_{AC}F_{AB}\cos 60^\circ\right)}$$

$$= \sqrt{\left[\left(\frac{Gmm}{a^2}\right)^2 + \left(\frac{Gmm}{a^2}\right)^2 + 2\left(\frac{Gmm}{a^2}\right)\left(\frac{Gmm}{a^2}\right)\cdot\frac{1}{2}\right]}$$

$$= \frac{Gmm}{a^2}\left[\sqrt{3}\right].$$

This force provides the necessary centripetal force, i.e.,

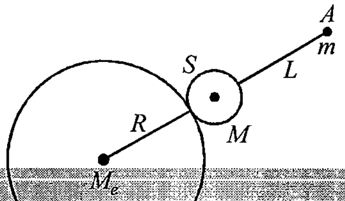
$$\frac{mv^2}{r} = \frac{Gmm}{a^2}\sqrt{3} \text{ or } \frac{mv^2}{\left(a/\sqrt{3}\right)} = \frac{Gmm}{a^2}\sqrt{3}$$

Therefore, required initial speed $v = \left\{\dfrac{Gm}{a}\right\}^{1/2}$

Time period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi\left(a/\sqrt{2}\right)}{(Gm/a)^{1/2}} = \dfrac{2\pi a^{3/2}}{(3Gm)^{1/2}}$.

38. A cord of length 64 m is used to connect a 100 kg astronaut to spaceship whose mass is much larger than that of the astronaut. Estimate the value of the tension in the cord. Assume that the spaceship is orbiting near earth surface. Also assume that the spaceship and the astronaut fall on a straight line from the earth center. The radius of the earth is 6400 km.

**Soln.** See figure below. Given that $R = 6400$ km, $l = 64$ m and mass $m$ of astronaut $= 100$ kg.



The gravitational attraction between earth and spaceship $S = \dfrac{GM_eM}{R^2}$.

This force supplies the necessary centripetal force to spaceship, i.e.,

$$\frac{GM_eM}{R^2} = \frac{Mv^2}{R} = M\omega^2 R \text{ or } \omega^2 = \frac{GM_e}{R^3} \text{ or } \omega = \sqrt{\left(\frac{GM_e}{R^3}\right)}.$$

But, $\dfrac{GM_e}{R^2} = g$, hence $\omega = \sqrt{\left(\dfrac{g}{R}\right)}$ ... (i)

The centrifugal force on the astronaut $F_1 = m\omega^2(R+l)$ ... (ii)

The centripetal force on astronaut $F_2 = \dfrac{GM_em}{(R+l)^2}$ ... (iii)

The tension in the cord would be the difference of centrifugal force and centripetal force i.e., $T = F_1 - F_2$.

$$\therefore \quad T = m\omega^2(R+l) - \frac{GM_em}{(R+l)^2} \qquad \text{... (iv)}$$

Substituting the value of $\omega$ from equation (i) and $GM_e = gR^2$ in equation (iv), we get

$$T = \frac{mg(R+l)}{R} - \frac{gR^2m}{(R+l)^2} = mg + \frac{mgl}{R} - \frac{mg}{[1+(l/R)]^2} = mg + \frac{mgl}{R} - mg\left[1 - \frac{2l}{R}\right]$$

$$= mg + \frac{mgl}{R} - mg + \frac{2mgl}{R} = \frac{3mgl}{R} = \frac{3\times 100\times 10\times 64}{6400\times 10^3} = \frac{3}{10^2} = \mathbf{3\times 10^{-2}\ N}.$$

# Chapter 3

## *WORK ENERGY & POWER*

**Conservative force:** A force is said to be conservative if curl of the force is zero.

i.e. if $\vec{\nabla}\times\vec{F}=0$ then, $\vec{F}$ is conservative force.

Work done by a conservative force is independent of path followed and work done along a closed path is zero. Gravitational and electrostatic forces are conservative. Friction, viscous forces are not conservative.

**Potential energy:** It is defined as work done by conservative force in taking the system from given configuration (position) to a reference position. Reference positions is that where the conservative force vanishes.

Potential energy can be directly obtained from relation

$$U=-\int F dr \text{ or } U=-\int F dx$$

for attractive force F is used with negative sign and for repulsive force F is used with positive sign.

**Spring Force:** When a spring is either stretched or compressed such that its length changes by $x$ then restoring force or tension in the spring is $-kx$.

**Spring potential energy:** Particle attached to the spring is always attracted towards mean position. Therefore, $F=-kx$

$$\therefore U=-\int(-kx)dx=\frac{1}{2}kx^2$$

**Gravitational potential energy for two particles:**

We know that gravitational force between two particles of masses $m_1$ and $m_2$ at separation $r$ is always attractive and is equal to

$$f=-\frac{Gm_1.m_2}{r^2} \qquad \therefore U=-\int\left(-\frac{Gm_1.m_2}{r^2}\right)dr=-\frac{Gm_1.m_2}{r}$$

**Work:** It is defined as $W=\int\vec{F}.\overrightarrow{dl}$

where $\overrightarrow{dl}$ is infinitesimal displacement of the object on which force is acting.

**Power:** It is defined as dot product of force and instantaneous velocity $P=\vec{F}.\vec{v}$

Average power is defined as total work done divided by total time $\langle P\rangle=\frac{W}{t}$

**Kinetic Energy:** For a particle of mass '$m$' and speed v, kinetic energy is defined as $K=\frac{1}{2}mv^2$

In terms of momentum it is written as $K=\frac{p^2}{2m}$.

- **Work-energy theorem:** It states that total work done by all forces is equal to change in kinetic energy of system.

$$W = K_{final} - K_{initial}$$

**Conservation of mechanical energy:** If only conservative forces are acting on a system then total mechanical energy of system remains conserved. Therefore

$$E_{initial} = E_{final}$$

$$\text{or} \quad K_i + U_i = K_f + U_f \;\Rightarrow\; U_i - U_f = K_f - K_i$$

Therefore, loss in potential energy = gain in kinetic energy (or vice versa).

## SOLVED EXAMPLES

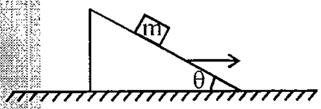
1. A particle moves from point (1, –1, 2) to (2, 3, 5) when a force $\vec{F} = 3\hat{i} + 5\hat{j} + 4\hat{k}$ acts on it. Calculate work done by the force.

**Soln.** The given force is constant. Therefore it is a conservative force. Therefore work done is independent of path followed.

$$W = \int \vec{F}.d\vec{l} = \int F_x.dx + \int F_y dy + \int F_z dz = \int_1^2 3dx + \int_{-1}^{3} 5dy + \int_2^5 5dz$$

$$= 3(2 - 1) + 5(3 + 1) + 5(5 - 2) = 38 \text{ Joule}$$

2. A block of mass $m$ is lying on a wedge as shown in figure. The block does not slide on the wedge while the wedge moves horizontally with uniform speed. Calculate work done by normal reaction on the block during the time wedge moves by a distance S in horizontal direction.

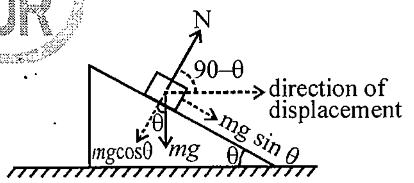


**Soln.** From the figure shown we find that $N = mg\cos\theta$ and angle between N and direction of displacement is $90 - \theta$. Therefore work done by normal reaction is,

$$W = \int \vec{F}.d\vec{l} = \vec{N}.\int d\vec{l} = \vec{N}.\vec{S}$$

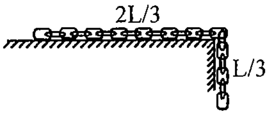
$$= N\,S\cos(90 - \theta) = mg\cos\theta.S\sin\theta$$

$$= mg\,S\sin\theta.\cos\theta$$



3. Two third length of a thin metallic chain of mass per unit length λ is held on a smooth table and remaining hanging off from its edge. If the chain is released what will be its speed at the moment the chain completely slips off the table.

**Soln.** There is no dissipative force acting on the system. Therefore total energy of the system remains conserved. Therefore we will use conservation of energy to find speed. Let us take potential energy to be zero at level of table top. Initially centre of mass of hanging part is L/6 below the table top and finally centre of mass of whole chain is L/2 below the table top. Therefore,



$$U_{initial} = -\left(\frac{\lambda L}{3}\right) g \left(\frac{L}{6}\right) = -\frac{\lambda g L^2}{18}, \quad U_{final} = -(\lambda L)\, g\left(\frac{L}{2}\right) = -\frac{\lambda g L^2}{2}$$

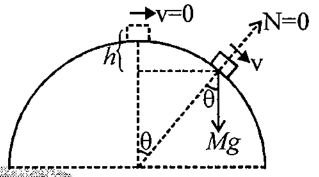
$$K_{initial} = 0,\ \ K_{final} = \frac{1}{2}(\lambda L)v^2$$

From conservation of energy $K_{initial} + U_{initial} = K_{final} + U_{final}$

$$0+\left(-\frac{\lambda g L^2}{18}\right)=\frac{1}{2}(\lambda L)v^2+\left(-\frac{\lambda g L^2}{2}\right) \quad \Rightarrow v=\sqrt{\frac{8gL}{9}}$$

4. A small object is released at the top of a smooth fixed sphere due to which it slides on the sphere for some time and then breaks off (leaves) the sphere. Calculate the angle with upward vertical at the position where object leaves the sphere.

**Soln.** Let the angle with upward vertical is $\theta$. Therefore height descended by object from its initial position.



$h = R(1-\cos\theta)$

from conservation of energy

loss in potential energy = gain in kinetic energy

$$mgR(1-\cos\theta)=\frac{1}{2}mv^2$$

$$\therefore\ v^2 = 2Rg(1-\cos\theta) \qquad \text{... (a)}$$

At the moment of leaving the sphere the object is still moving on circular path of radius R. Therefore

Net force toward centre = $\dfrac{mv^2}{R}$, or $mg\cos\theta = \dfrac{mv^2}{R}$

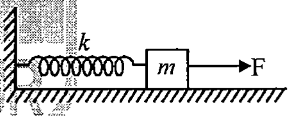
or $v^2 = Rg\cos\theta$ ... (b)

from (a) and (b) we get $\cos\theta = \dfrac{2}{3}$ or $\theta = \cos^{-1}\dfrac{2}{3}$

5. A block of mass $m$ attached to a light spring of spring constant $k$ lies in on a smooth horizontal plane as shown in figure. A horizontal force F is applied to the block.

Calculate maximum speed of the block.

Also calculate maximum elongation in spring.



**Soln.** When force is applied, the block accelerates for some time then it decelerates. Therefore acceleration of block is initially positive, then it momentarily becomes zero and finally becomes negative till the block stops. Speed of block is maximum at the instant when its acceleration is zero.

$$a_x = 0 \Rightarrow F_x = 0 \qquad \therefore\ F - kx = 0 \Rightarrow x = \frac{F}{k}$$

Thus speed is maximum when elongation in spring is equal to $\dfrac{F}{k}$

To find maximum speed let us use work energy theorem.

Total work done = change in kinetic energy

$$W_{Spring} + W_{Force} = \frac{1}{2}mv^2 - 0 \quad \text{or} \quad \int_0^{F/k} kx\,dx\cos 180^\circ + \int_0^{F/k} F\,dx = \frac{1}{2}mv^2$$

$$-\frac{F^2}{2k}+\frac{F^2}{k}=\frac{1}{2}mv^2 \quad \text{or} \quad v=\sqrt{\frac{F^2}{mk}}$$

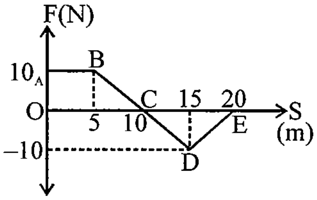
Let $x_0$ be the maximum elongation

When elongation becomes maximum the block again comes to rest.
Therefore from work energy theorem we get,

$$W_{Spring} + W_{Force} = 0 - 0, \quad \int_0^{x_0} kxdx\cos180^\circ + \int_0^{x_0} Fdx = 0$$

$$-\frac{1}{2}kx_0^2 + Fx_0 = 0, \qquad \therefore x_0 = \frac{2F}{k}$$

6. Force acting on a particle of mass 2kg varies with distance as shown in the figure. If speed of the particle at S = 0m be 5 m/s, calculate its speed at S = 20 m.



**Soln.** Total work done

= area OABCO − area CEDC

$$= 10\times5+\frac{1}{2}\times10\times5 - \frac{1}{2}\times10\times10 = 75 - 50 = 25 \text{ Joule}$$

Work energy theorem

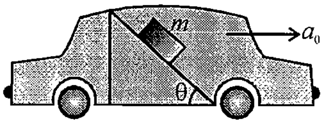
Total work done = change in K.E.

$$25 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \quad = \frac{1}{2}\times2\times v_2^2 - \frac{1}{2}\times2\times25$$

$$v_2^2 = 50 \quad \Rightarrow v_2 = 5\sqrt{2} \text{ m/s}$$

7. A block of mass $m$ stays at rest on an inclined plane inside a car which is moving with constant acceleration $a_0$. Calculate work done by friction on block as the car moves by a distance $x$.

**Soln.** Free body diagram of block inside the car is shown in figure.

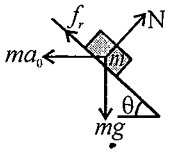


Block is in equilibrium.

∴ Along the plane

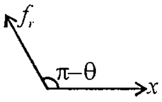
$$f_r + ma_0\cos\theta = mg\sin\theta$$

or $f_r = m(g\sin\theta - a_0\cos\theta)$



As the car moves through a distance '$x$', the block also moves the same distance.

∴ Work done by friction

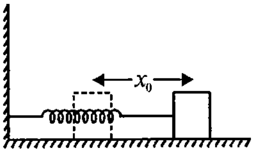


$$= \vec{f}_r\cdot\vec{x} \quad = f_r\,x\cos(\pi-\theta)$$

$$= -f_r\,x\cos\theta \quad = -mx(g\sin\theta - a_0\cos\theta)\cos\theta$$

8. A block of mass M is attached to one end of a spring of mass $m$ and force constant K. The other end of spring is attached to a wall and the block lies on a smooth horizontal surface. The block is compressed against the spring through a distance $x_0$ and released. Calculate the speed of the block when it passes through the mean position.

**Soln.** Let the spring is uniformly stretched



$\therefore$ displacement of an element $dx$

which is at a distance $x$ from the fixed end is

$$= \frac{x_0}{L} x; L \text{ is present length of spring.}$$

Therefore, if $v$ be the speed of block at an instant then velocity of element $dx$.

$$= \frac{v}{L} x$$

$$\therefore \text{ K.E. of element } = \frac{1}{2} \cdot \left( \frac{m}{L} dx \right) \cdot \left( \frac{v}{L} x \right)^2$$

$$\therefore \text{ K.E. of spring } = \int_0^L \frac{1}{2} \left( \frac{m}{L} dx \right) \left( \frac{v}{L} x \right)^2 = \frac{1}{6} mv^2$$

Now, let $v_0$ be the speed of block when it passes through the origin.

$\therefore$ From conservation of energy we get, $\frac{1}{2} kx_0^2 = \frac{1}{2} Mv_0^2 + \frac{1}{6} mv_0^2$

$$\therefore \quad v_0 = x_0 \sqrt{\frac{3K}{3M + m}}$$

9. A man is drawing water from a well with a bucket which leaks uniformly. The bucket when full weights 20 kg and when it arrives the top only half the water remains. The depth of the water is 20 meters. If $g = 10$ m/sec$^2$, what is the work done ?

**Soln.** When the bucket arrives at the top, the mass is 10 kg. Hence loss in mass = 20 – 10 = 10 kg. The depth of the well is 20 meters.

Therefore, mass lost per unit distance $= \frac{10}{20} = \frac{1}{2}$ kg.

Consider a point at a height $x$ from the bottom of the well.

At height $x$ from the bottom, the bucket weight $= \left( 20 - \frac{x}{2} \right)$ kg. The workdone against the force during elementary displacement $dx = \left( 20 - \frac{x}{2} \right) dx \cdot g$.

Therefore, total workdone

$$W = \int_0^{20} \left( 20 - \frac{x}{2} \right) g\, dx = g \int_0^{20} \left[ 20\, dx - \frac{1}{2} x\, dx \right] = g \left[ \{ 20x \}_0^{20} - \left\{ \frac{x^2}{4} \right\}_0^{20} \right]$$

$$= g[400 - 100] = 300g = 300 \times 10 = \mathbf{3000\ J.}$$

10. A body of mass $m$ is thrown at an angle $\alpha$ to the horizontal with an initial velocity $v_0$. Find the mean power developed by gravity over the whole time of motion of the body and the instantaneous power of gravity as a function of time.

**Soln.** We know that $P_{\text{instan.}} = F \cdot v$

The velocity of the particle after time $t$ is given by $v = v_0 \cos\alpha\, i + (v_0 \sin\alpha - gt)\, j$ and $F = -mg\, j$.

Therefore, $P_{\text{instan.}} = (-mg\, j)\cdot\{v_0 \cos\alpha\, i + (v_0 \sin\alpha - gt)\, j\} = -mg\,(v_0 \sin\alpha - gt)$.

The average power is given by

$$\langle P\rangle = \frac{\int_0^T P(t)\,dt}{\int_0^T dt} = \frac{-mg\int_0^T (v_0 \sin\alpha - gt)\,dt}{T} = \frac{-mg\left[v_0 \sin\alpha T - gT^2/2\right]}{T}.$$

Here $T$ is total of flight $T = \frac{(2v_0 \sin\alpha)}{g}$.

Substituting this value, we have

$$\langle P\rangle = -mg\left[\frac{v_0 \sin\alpha\,(2v_0 \sin\alpha)/g - g\,(4v_0^2 \sin^2\alpha)/g^2}{(2v_0 \sin\alpha)/g}\right]$$

Solving we get $\langle P\rangle = 0$.

11. A particle of mass $m$ moves along a circle of radius $R$ with a normal acceleration varying with time as $a_n = kt^2$, where $k$ is a constant. Find time dependence of power developed by all the forces acting on the particle and the mean value of this power averaged over the first $t$ seconds after the beginning of the motion.

**Soln.** Normal acceleration $a_n = kt^2$

Therefore, $\frac{v^2}{R} = kt^2$ or $v^2 = kRt^2$ or $v = \sqrt{(kR)}\, t$ or $\left(\frac{dv}{dt}\right) = \sqrt{(kR)}$

We know that in circular motion, workdone by normal force is zero. For tangential force

$$F_t = m\left(\frac{dv}{dt}\right) = m\sqrt{(kR)}$$

Now power $P = F_t \cdot v = F_t \cdot v\cos\theta = F_t\, v \quad (\because\ \theta = 0)$

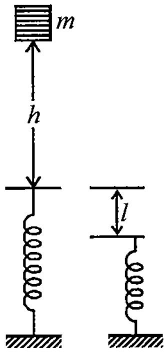
$$= mkRt$$

$$\text{Further, } \langle P\rangle = \frac{\int_0^T P(t)\,dt}{\int_0^T dt} = \frac{\int_0^T mkRt\,dt}{T} = \frac{mkRt\left[\frac{t^2}{2}\right]_0^T}{T} = \frac{mkRT}{2}.$$

12. A 2.0 kg block of mass, initially at rest, is dropped from a height of 0.40 meter onto a spring whose force constant is 1960 nt/meter. Find the maximum distance that the spring will be compressed.

**Soln.** The situation is shown in figure. Let $m$ be the mass of the block and $k$ be the force constant of the spring. Let $l$ be the distance through which the spring is compressed. The total vertical fall of the block is $(h+l)$.

Loss of the gravitational potential energy of the block $= mg\,(h+l)$.



Elastic potential energy in the spring $= \left(\frac{1}{2}\right)kt^2$.

By the law of conservation of energy

$$mg\,(h+l) = \frac{1}{2}kt^2 \quad \text{or} \quad \frac{2mgh}{k} + \frac{2mg}{k}l = l^2 \quad \text{or} \quad l^2 - \frac{2mg}{k}l - \frac{2mgh}{k} = 0$$

$$\therefore \quad l = \frac{\left(\frac{2mg}{k}\right) \pm \sqrt{\left[\left(\frac{2mg}{k}\right)^2 + \left(\frac{8mgh}{k}\right)\right]}}{2}$$

According to given problem, $m = 2$ kg, $h = 0.40$ m and $k = 1960$ newton/meter. Hence,

$$l = \frac{(2\times 2\times 9.8/1960) \pm \sqrt{(2\times 2\times 9.8/1960)^2 + (8\times 2\times 9.8\times 0.40/1960)}}{2}$$

$l = \mathbf{0.1}$ **meter.**

13. A particle of mass 3 kg is moving under the action of a central force whose potential energy is given by $U(r) = 10r^3$ joule. For what energy and angular momentum will the orbit be a circle of radius 10 m ? Calculate the time period of this motion.

**Soln.** Given that $U(r) = 10r^3$.

So the force $F$ acting on the particle is given by

$$F = \frac{\partial U}{\partial r} = -\frac{\partial}{\partial r}(10r^3) = -10\times 3r^2 = -30r^2$$

For circular motion of the particle $F = \frac{mv^2}{r} = 30r^2$.

Substituting the given values, we have $\frac{3\times v^2}{10} = 30\times(10)^2$ or $v = 100$ m/s.

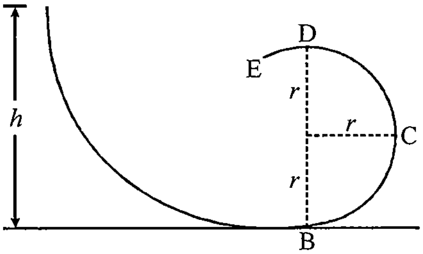
The total energy in circular motion

$$E = \text{K.E.} + \text{P.E.} = \frac{1}{2}mv^2 + U(r) = \frac{1}{2}\times 3\times(100)^2 + 10\times(10)^3 = 2.5\times 10^4 \text{ joule}$$

Angular momentum $= mvr = 3\times 100\times 10 = 3000\ \text{kg-m}^2/\text{sec}$

Time period $T = \dfrac{2\pi r}{v} = \dfrac{2\times\pi\times 10}{100} = \dfrac{\pi}{5}\ \text{sec}.$

14. In figure, $ABCDE$ is a channel in the vertical plane, part $BCDE$ being circular with radius $r$. A ball is released from $A$ and slides without friction and without rolling. Show that it will complete the loop path if $h$ is greater than $5r/2$.



**Soln.** Let $m$ be the mass of the ball. When the ball comes down to $B$, its loses its potential energy $mgh$ which is converted into kinetic energy. Let $v_B$, be the velocity of the ball at $B$. Then,

$$mgh = \frac{1}{2}mv_B^2 \qquad \text{...(i)}$$

The ball now rises to a point $D$, where its potential energy is $mg(h-2r)$. If $v_D$ be the velocity of the ball at $D$, then

$$mg(h-2r) = \frac{1}{2}mv_D^2 \qquad \text{...(ii)}$$

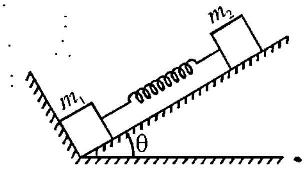
Now to complete the circular path, it is necessary that the centripetal force acting upward at point $D$ should be equal or greater than the force $mg$ acting downward. Therefore,

$$\frac{mv_D^2}{r} \geq mg \ \text{ or } \ v_D^2 \geq rg \qquad \text{...(iii)}$$

From equation (ii) $v_D^2 = 2g(h-2r)$

$$\therefore \quad 2g(h-2r) \geq rg,\ h \geq \frac{5}{2}r.$$

15. Two blocks of masses $m_1$ and $m_2$ attached to the ends of a spring of spring constant K, lie on a smooth inclined surface as shown in the figure. Find initial compression in the spring so that $m_1$ just leaves the contact with the left wall (take $m_1 = m_2 = m$)



**Soln.** $m_1$ will leave contact because the spring will elongate and $m_2$ will rise up.

$\therefore \quad m_1 g\sin\theta = kx$

where $x$ is elongation

$$\therefore \quad x = \frac{m_1 g\sin\theta}{k}$$

Let $x_0$ be initial compression in spring

$\therefore \quad m_2$ rises up the plane by a distance of $x_0 + x$

$\therefore$ from conservation of energy we get

$$\frac{1}{2}kx_0^2 = m_2g(x+x_0)\sin\theta + \frac{1}{2}kx^2 = m_2g(x+x_0)\sin\theta + \frac{1}{2}k.\frac{m_1^2g^2\sin^2\theta}{k^2}$$

$$= \frac{m_2g\cdot m_1g\sin^2\theta}{k} + m_2g\,x_0\sin\theta + \frac{m_1^2g^2\sin^2\theta}{2k}$$

$$\frac{1}{2}kx_0^2 = \frac{m_1\left(\frac{m_1}{2}+m_2\right)g^2\sin^2\theta}{k} + m_2g\,x_0\sin\theta$$

$$\therefore \quad x_0^2 - \left(\frac{2m_2g\sin\theta}{k}\right)x_0 - \frac{m_1(m_1+2m_2)g^2\sin^2\theta}{k^2}$$

for $m_1 = m_2 = m$

$$x_0^2 - \left(\frac{2mg\sin\theta}{k}\right)x_0 - \frac{3m^2g^2\sin^2\theta}{k^2} = 0$$

solving this quadratic equation we get

$$x_0 = \frac{3mg\sin\theta}{k}$$

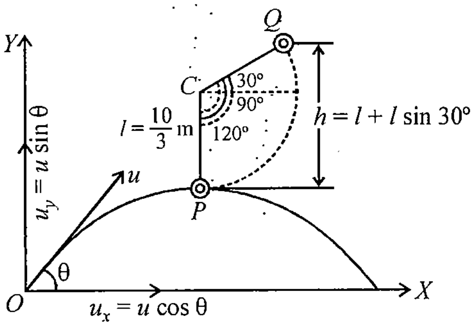
16. A bullet of mass $M$ is fired with a velocity of 50 m/sec. at an angle θ with the horizontal. At the highest point of its trajectory, it collides head on with a bob of mass $3M$ suspended by a massless string of length 10/3 metres and gets embedded in the bob. After the collides, the string moves through an angle of 120°. Find (i) the angle θ; (ii) the vertical and horizontal co-ordinates of the initial position of the bob with respect to the point of firing of the bullet (take g = 10 m/sec²).

**Soln.** (i) The situation is shown in figure. At highest point, the vertical components becomes zero, while the horizontal component remains unchanged, i.e., $u_y = 0$ and $u_x = u\cos\theta$.

Now applying the law of conservation of linear momentum, we have

$$M\,u_x = (M+3M)v, \qquad \text{...(i)}$$

where $v$ is the final velocity.



According to particle of conservation of energy

$$\frac{1}{2}(M+2M)v^2 = (M+3M)\,g\,h$$

or $\quad v^2 = 2g\,h = 2g(l + l\sin 30). \qquad$ ... (ii)

From equation (i),

$$v = \frac{u_x}{4} = \frac{u\cos\theta}{4} = \frac{50\cos\theta}{4}$$

Substituting this value of $v$ in equation (ii), we get

$$\left(\frac{50\cos\theta}{4}\right)^2 = 2g\,(l + l\sin 30^\circ)$$

or $$\cos^2\theta = 2gl(1+\sin 30^\circ)\left(\frac{4}{50}\right)^2 = 2\times10\times\left(\frac{10}{3}\right)\left(\frac{4}{50}\right)^2(1+\sin 30^\circ)$$

$$= 2\times10\times\left(\frac{10}{3}\right)\left(\frac{4}{50}\right)^2\left(1+\frac{1}{2}\right) = 100\times\left(\frac{4}{50}\right)^2$$

$\therefore$ $$\cos\theta = 10\times\left(\frac{4}{50}\right) = \left(\frac{4}{5}\right)$$

$$= \cos^{-1}\left(\frac{4}{5}\right) = \cos^{-1}(0.8) = 37^\circ.$$

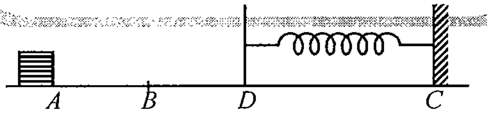
(ii) Time of ascent, $t = \dfrac{u\sin\theta}{g} = \dfrac{50\sin 37^\circ}{10} = 5\sin 37^\circ = 5\times0.6 = 3$ sec

Therefore, horizontal coordinate $x = u\cos\theta\cdot t = 50\times0.8\times3 = 120$ m

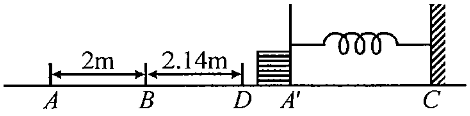
Vertical coordinate $y = u\sin\theta\cdot t - \dfrac{1}{2}gt^2 = 50\times0.6\times3 - \dfrac{1}{2}\times10\times9 = 45$ m.

17. A 0.5 kg block slides from the point $A$ [see figure (a)] on a horizontal track with an initial speed of 3 m/s towards a weightless horizontal spring of length 1 meter and force constant 2 Nm$^{-1}$. The part $AB$ of the track is frictionless and the part $BC$ has the coefficients of static and kinetic friction as 0.22 and 0.2 respectively. If the distance $AB$ and $BD$ are 2 m and 2.14 m respectively, find the total distance through which the block moves before it comes to rest completely. Take $g = 10$ m/s$^2$.

**Soln.** The situation is shown in figure (b).



(a)



(b)

First of all the block moves the track $AB$ without change in kinetic energy as it is frictionless. The initial kinetic energy of the block is given by $\frac{1}{2}mv^2$.

Therefore, initial kinetic energy of block $= \frac{1}{2}\times0.5\times3^2 = 2.25$ joule.

Now the block enters the track $BC$. This track has a coefficient of kinetic friction as 0.2. A part of the kinetic energy is used in workdone against frictional force. The workdone against friction over the distance $BC$ is given by $= F \times s = \mu_k m g s = 0.2 \times 0.5 \times 10 \times 2.14$ joule.

This workdone is less than the initial kinetic energy of the block, hence the block moves further and compresses the spring by a distance say $x$. Hence, the initial kinetic energy of the block = workdone against the friction of track $BD$ + workdone against the friction of distance $x$ + potential energy of the compressed spring

$$\therefore \quad 2.25 = 2.14 + 0.2 \times 0.5 \times 10 \times x + \frac{1}{2}kx^2$$

where $k$ is the force constant of the spring or $2.25 = 2.14 + x + x^2$

$$x^2 + x - 0.11 = 0 \qquad x = 0.1\text{m} \text{ or } (-1.1\text{m})$$

As the block compresses the spring, $x$ can not be negative. Thus $x = 0.1$ meter

Moving through a distance of 0.1 m, the block comes to rest. The compressed spring now exerts a force on the block given by $F = kx = 2 \times 0.1$ newton.

The limiting value of the frictional force between block and track

$$BC = \mu_s \, mg = 0.22 \times 0.5 \times 10 = 1.1 \text{ newton.}$$

Since the limiting frictional force on the block by track is more than the force applied by the spring, the block comes to rest completely.

Total distances moved by the block,

$$= AB + BD + x$$

$$= 2 + 2.14 + 0.1 = \mathbf{4.24\ metre.}$$

18. A small ring of mass $m$ can slide on a smooth circular wire of radius $r$ and center $O$, which is fixed in a vertical plane. From a point on the wire at a vertical distance $r/2$ above $O$, the ring is given a velocity $\sqrt{(gr)}$ along the downward tangent to the wire. Show that it will just reach the highest point of the wire. Find the reaction between the ring and the wire when the ring is at a vertical distance $r/2$ below.

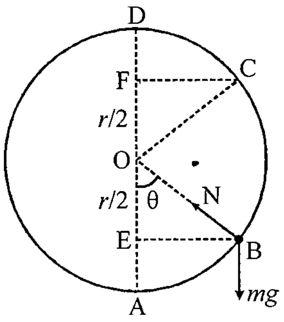
**Soln.** At point $C$, the velocity of the ring $= \sqrt{(rg)}$.

K.E. at $C = \frac{1}{2}mv^2 = \frac{1}{2}mrg$ and P.E. at $C = mgh = mg(AF) = mg(r + r/2) = \frac{3mgr}{2}$.

$$\text{Total energy} = mgr\left[\frac{1}{2} + \frac{3}{2}\right] = 2mgr. \qquad \text{...(i)}$$

If the ring has to reach at $D$, with kinetic energy zero, its potential energy $= mg(2r)$.

Becuase the particle at $C$ has this amount of energy and it will just reach at $D$ with zero velocity.



Now consider the ring at $B$, a distance $r/2$ below $O$. Resolving $mg$ in two parts and considering the equilibrium, we get

$$N - mg\cos\theta = \frac{mv_1^2}{r}$$

or $$N = \frac{mv_1^2}{r} + mg\cos\theta \qquad \text{... (ii)}$$

Falling from *C*, i.e., through a distance *r*, the ring has lost a potential energy $mgr$. This is the gain in kinetic energy.

Hence, $\frac{1}{2}mv_1^2 = \frac{1}{2}mgr + mgr = \frac{3}{2}mgr$ or $v_1^2 = 3gr$ or $\frac{v_1^2}{r} = 3g$.

From equation (ii), $N = m(3g + g \times \frac{1}{2})$ $\qquad (\because \cos\theta = \frac{1}{2})$

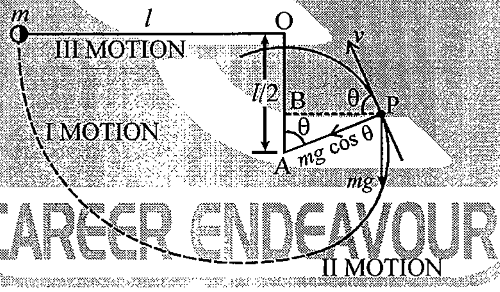
or $N = \left(\frac{7}{2}\right)mg = 3.5\,mg$.

19. A small ball is suspended from point *O* by a thread of length *l*. A nail is driven into the wall at a distance of *l*/2 below *O*, at *A*. The ball is drawn aside so that the thread takes up a horizontal position. At what point to the ball trajectory, will the tension in the thread disappear ? How much further will the ball move ? What will be the highest point to which it will rise ?

**Soln.** The ball will perform the following :

(i) Firstly, the ball describes a quadrant of circle of radius equal to the length of the thread.

(ii) Secondly, the thread touches the nail and describes a circular motion till the tension in thread becomes zero [point *P* in figure below].



(iii) Thirdly, the ball continuous its path just like a body thrown at certain angle (θ in figure) with certain initial velocity. When the tension in the thread becomes zero at *P*, the component of mass along the radius provides centripetal force,

i.e., $$mg\cos\theta = \frac{mv^2}{r} \qquad \text{... (i)}$$

where $r = \frac{l}{2}$.

To calculate the velocity of ball at *P*, we proceed as follows:

Kinetic energy at *P* = Decrease in potential energy due to a distance *OB*, $\frac{1}{2}mv^2 = mg\,(OB)$.

From figure, $OB = OA - AB$.

Now, $OB = \frac{l}{2} - \frac{l}{2}\cos\theta$.

Therefore, $\frac{1}{2}mv^2 = mg\left(\frac{l}{2}-\frac{l}{2}\cos\theta\right)$ or $v^2 = gl(1-\cos\theta)$ ... (ii)

From equation (i), $mg\cos\theta = \frac{mgl(1-\cos\theta)}{l/2}\cos\theta = \frac{2}{3}$.

Therefore, the tension in the thread becomes zero at height, $\frac{l}{2}+\frac{l\cos\theta}{2} = \frac{l}{2}+\frac{l}{2}\times\frac{2}{3} = \frac{5l}{6}$ from lowest point.

From equation (ii), $v^2 = gl\left(1-\frac{2}{3}\right) = \frac{gl}{3}$ or $v = \sqrt{\left(\frac{gl}{3}\right)}$.

Now, the particle describes a parabolic path. The height point lies above the point of throwing at a distance, $\frac{(v\sin\theta)^2}{2g} = \frac{(gl/3)\cdot(5/9)}{2g} = \frac{5l}{54}$.

# Chapter 4

## MOMENTUM, COLLISION & CENTRE OF MASS

**Conservation of momentum:** Total momentum of a system remains conserved if there is no external force on the system. This conclusion comes directly from Newton's second law $\vec{F} = \frac{d\vec{p}}{dt}$, if $\vec{F} = 0$, $\frac{d\vec{p}}{dt} = 0$

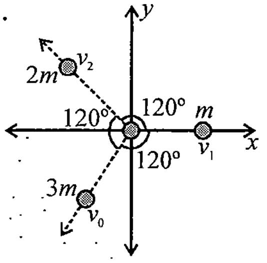
$\Rightarrow \vec{p} = \text{constant}$

**Note:** If two particles are interacting with each other, momentum of each particle changes with time, however sum of momenta of the two particles remains constant. For example in rocket propulsion, momentum of rocket increases with time, however sum of momenta of fuel and rocket remains constant.

**Example:** An object initially kept at rest breaks up into three fragments of mass ratio 1 : 2 : 3. If speed of heaviest fragment just after explosion is $v_0$ what is speed of lightest fragment. It is given that the three fragments move in same plane and make equal angle with each other.

**Soln.** There is no external force on the object. Therefore its momentum remains constant. Initial momentum is zero. Therefore vector sum of momenta of three fragments must be zero.

$\vec{p} = 0 \Rightarrow p_x = 0, p_y = 0$. Let the masses of three fragments be $m$, $2m$ and $3m$.



Let $v_1$ and $v_2$ be speed of fragments of masses $m$ and $2m$.

It is given that speed of $3m$ is $v_0$.

$p_x = 0$

$\Rightarrow \; mv_1 - 2mv_2 \cos 60 - 3mv_0 \cos 60 = 0$

$$\therefore \; v_1 - v_2 = \frac{3v_0}{2} \qquad \text{... (a)}$$

$$p_y = 0 \qquad \Rightarrow \; 2mv_2 \sin 60 - 3mv_0 \sin 60 = 0$$

$$\therefore \; v_2 = \frac{3v_0}{2} \qquad \text{... (b)}$$

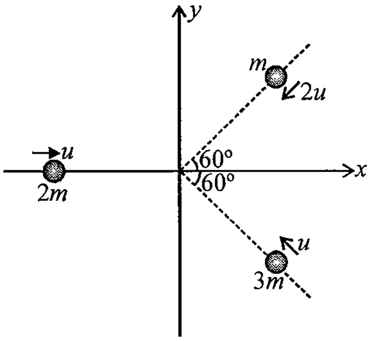
from (a) $v_1 = \frac{3v_0}{2} + v_2 = \frac{3v_0}{2} + \frac{3v_0}{2} = 3v_0$

Therefore speed of the lightest fragment is $3v_0$.

**Example :** Three particles moving as shown in figure, collide at origin and coalesce to form a single particle. Calculate speed of the new particle formed.

**Soln.** There is no external force on the system of three particles. Therefore momentum must be conserved or each component of moment must be conserved.

Let $V_x$ and $V_y$ be $x$ and $y$ components of velocity of new particle



Therefore, $p_{x_{initial}} = p_{x_{final}}$

$$2mu - m2u\cos 60 - 3mu\cos 60 = 6mV_x$$

$$\therefore V_x = -\frac{u}{12} \text{ and } p_{y_{initial}} = p_{y_{final}}$$

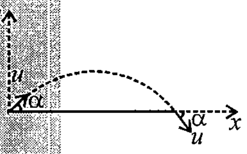
$$-m2u\sin 60 + 3mu\sin 60 = 6mV_y \qquad \therefore V_y = \frac{\sqrt{3}u}{12}$$

Therefore, velocity of new particle $= \sqrt{V_x^2 + V_y^2} = \frac{u}{12}\sqrt{3+1} = \frac{u}{6}$

Short time interaction between objects when they come in contact is called collision. If the colliding objects are moving with small speeds then collision can be studied using non-relativistic mechanics. However, if objects are moving with the large speed then relativitics mechanics is required. Here we will discuss collision at small speeds only.

**Impulse:** It is equal to change in momentum of object $\vec{I} = \vec{p}_{Final} - \vec{p}_{Initial}$ or $\vec{I} = \int \vec{F}dt$

**Example:** A projectile is thrown with initial velocity $u$ at angle $\alpha$ with horizontal. Calculate total impulse on the projectile due to gravitational force:



**Soln.** Initial momentum of projectile $\vec{p}_{initial} = mu(\cos\alpha\hat{i} + \sin\alpha\hat{j})$

final momentum of projectile $\vec{p}_{final} = mu(\cos\alpha\hat{i} - \sin\alpha\hat{j})$

Impulse on the projectile, $\vec{I} = \vec{p}_{final} - \vec{p}_{initial} = -2mu\sin\alpha\hat{j}$

**Types of Collision:**

1. **Elastic:** Internal energy of colliding objects doesn't change. It implies that kinetic energy is not converted into internal energy. Therefore, in cae of elastic collision, we have following conservation laws

(a) Conservation of kinetic energy: Sum of kinetic energy of all colliding objects remains constant.

i.e. $\text{K.E.}_{\text{initial}} = \text{K.E}_{\text{final}}$

(b) Convervation of momentum: Sum of momentum, of all colliding objects remains constant.

i.e. $\vec{p}_{initial} = \vec{p}_{final}$

or, $(p_{initial})_x = (p_{final})_x, (p_{initial})_y = (p_{final})_y$

2. **Inelastic:** Internal energy of colliding objects changes due to which kinetic energy doesn't remain conserved. Therefore, in case of inelastic collision we can only apply conservation of momentum. **Pefectly inelastic collision** is a special case of inelastic collision in which objects stick to each other.

**How to distinguish elastic and inelastic collision:**

If collision is elastic then the in the problem it is explicitly mentioned in most cases that collision is elastic. However, if collision is inelastic then it may be explicitly mentioned or phrases such as 'there is loss of kinetic energy' or 'breaking of objects' or 'objects stick to each other' etc are used.

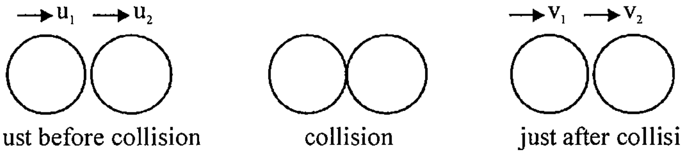
**Coefficient of restitution:**

It is defined as

$$e = \frac{|\text{velocity of separation after collision}|}{|\text{velocity of approach before collision}|}$$

For elastic collision (involving no rotation) $e = 1$. For inelastic collision $0 \le e \le 1$. If objects stick to each other after collision. Then $e = 0$

**Elastic collision in one dimension:**
Let $m_1$ and $m_2$ be masses of two objects having velocities $u_1$ and $u_2$ just before collision and velocities $v_1$ and $v_2$ just after collision.



From conservation of momentum we get,

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \qquad \text{... (i)}$$

From conservation of kinetic energy we get,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + m_2v_2^2 \qquad \text{... (ii)}$$

Solving (i) and (ii), we get

$$\boxed{\begin{aligned} v_1 &= \frac{u_1(m_1 - m_2) + 2m_2u_2}{m_1 + m_2} \\ v_2 &= \frac{u_2(m_2 - m_1) + 2m_1u_1}{m_1 + m_2} \end{aligned}}$$

Using above results we can calculate velocities of objects after collision in one dimensional elastic collision if velocities before collision is either given or can be determined by some method. The above results have been derived by assuming that objects are moving in same direction. If in a problem objects are moving in opposite direction then one velocity must be taken to be negative.

**Example:** Two objects of masses m and 4m are moving with speed u and 2u in opposite direction. What is their velocities after collision

**Soln.** Let us take, $m_1 = m$, $u_1 = u$ and $m_2 = 4m$, $u_2 = -2u$

Therefore, velocity of $m$ after collision

$$v_1 = \frac{u_1(m_1 - m_2) + 2m_2u_2}{m_1 + m_2} = \frac{u(m - 4m) + 2(4m)(-2u)}{5m} = \frac{-19}{5}u$$

Velocity of 4m after collision,

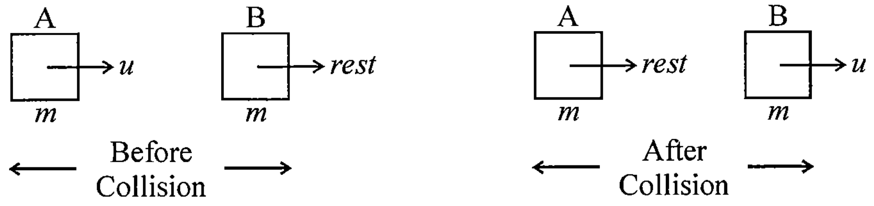
$$v_2 = \frac{u_2(m_2 - m_1) + 2m_1u_1}{m_1 + m_2} = \frac{-2u(4m - m) + 2mu}{5m} = -\frac{4}{5}u$$

Therefore, after collision both the masses move in the direction of velocity of 4m before collision.

**Special case**

- I-d Elastic collision of two equal masses

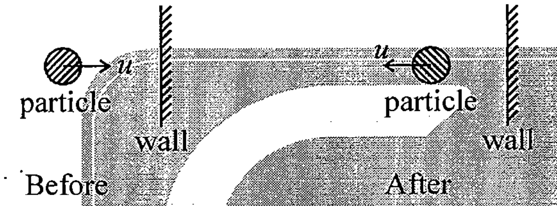
If $m_1 = m_2$ then, $v_1 = u_2$ and $v_2 = u_1$ i.e. the two objects exchange their velocities.


A
B
u
rest
m
m
Before Collision
A
B
rest
u
m
m
After Collision

- One object is much heavier than the other object and heavier object is initially at rest.

$m_1 >> m_2$ and $u_1 = 0$

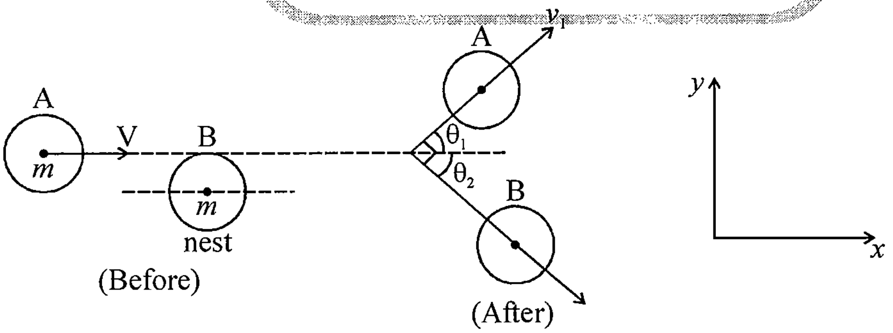
then, $v_1 = 0$ and $v_2 = -u_2$


u
particle
wall
Before
u
particle
wall
After

**Elastic collision in two dimension:**
In this case also we apply conservation of linear momentum and conservation of kinetic energy to solve problems. However momentum conservation (in most cases) has to be applied separately along two perpendicular directions (Viz X and Y directions).

**Special case:**
If elastic collision between two objects of equal masses is non-headon (not 1 dimensional) and one of them is initially at rest then after collision the two masses move at right angle.

Let us consider off center collision between two blocks A and B as shown in figure.


A
V
B
m
m
nest
(Before)
$v_1$
A
$\theta_1$
$\theta_2$
B
(After)
y
x

As shown in figure velocity vector $(\vec{v})$ of A before collision dosn't pass through center of B, therefore collision will definitely be two dimensional.

Let us consider off centre collision between two blocks A and B as shown in figure. As shown in figure velocity vector $(\vec{v})$ of vector A before collision doesn't pass through centre of B. Therefore, collision will definitely be two dimensional.

Let after collision A and B move with speed $v_1$ and $v_2$ and at angles $\theta_1$ and $\theta_2$ with direction of initial velocity $(\vec{v})$ of A.

Then as mentioned before $\theta_1 + \theta_2 = \frac{\pi}{2}$

**Proof:** From conservation of kinetic energy we get,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

or $\quad v^2 = v_1^{\,2} + v_2^{\,2}$ —(*i*)

From converstion of linear momentum in $x$-direction we get,

$$mv = mv_1\cos\theta_1 + mv_2\cos\theta_2$$

or $\quad v = v_1\cos\theta_1 + v_2\cos\theta_2$ —(*ii*)

from conservation of linear momentum in $y$ direction we get,

$$0 = mv_1\sin\theta_1 - mv_2\sin\theta_2$$

or $\quad v_1 = \dfrac{v_2\sin\theta_2}{\sin\theta_1}$ —(*iii*)

Put $v_1$ in (*ii*) to get,

$$v_2 = \frac{v\sin\theta_1}{\sin(\theta_1+\theta_2)}$$

use $v_2$ in (*iii*) to get,

$$v_1 = \frac{v\sin\theta_2}{\sin(\theta_1+\theta_2)}$$

Now put $v_1$ and $v_2$ in (*i*) to get,

$$\sin^2(\theta_1+\theta_2) = \sin^2\theta_1 + \sin^2\theta_2$$

or $\quad (\sin\theta_1\cdot\cos\theta_2 + \cos\theta_1\cdot\sin\theta_2)^2 = \sin^2\theta_1 + \sin^2\theta_2$

$$\sin^2\theta_1\cdot\cos^2\theta_2 + \cos^2\theta_1\cdot\sin^2\theta_2 + 2\sin\theta_1\cdot\sin\theta_2\cdot\cos\theta_1\cdot\cos\theta_2 = \sin^2\theta_1 + \sin^2\theta_2$$

or $\quad \sin^2\theta_1(1-\cos^2\theta_2) + \sin^2\theta_2(1-\cos^2\theta_1) - 2\sin\theta_1\cdot\sin\theta_2\cdot\cos\theta_1\cdot\cos\theta_2 = 0$

or $\quad 2\sin^2\theta_1\cdot\sin^2\theta_2 - 2\sin\theta_1\cdot\sin\theta_2\cdot\cos\theta_1\cdot\cos\theta_2 = 0$

or $\quad 2\sin\theta_1\cdot\sin\theta_2(\sin\theta_1\cdot\sin\theta_2 - \cos\theta_1\cdot\cos\theta_2) = 0$

Since $\theta_1 \neq 0,\ \ \theta_2 \neq 0$. Therefore

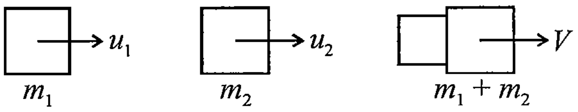
$$\sin\theta_1\cdot\sin\theta_2 - \cos\theta_1\cdot\cos\theta_2 = 0$$

or $\quad \cos(\theta_1+\theta_2) = 0$

Therefore, $\theta_1 + \theta_2 = \frac{\pi}{2}$

**Perfectly inelastic collision in one dimension:**

Consider the situation shown in figure in which two blocks moving with speed $u_1$ and $u_2$ collide and stick to each other.



Before After

From conservation of momentum we get,

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)V$$

Therefore, $$\boxed{V = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2}}$$

This is common speed of two block after collision
Kinetic energy is not conserved in the process.
so we can calculate loss in kinetic energy as

$$\Delta K = K \cdot E_i - K \cdot E_f$$

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2}(m_1 + m_2)V^2$$

$$= \frac{1}{2}\left[ m_1 u_2^2 + m_2 u_2^2 - (m_1 + m_2) \cdot \frac{(m_1 u_1 + m_2 u_2)^2}{(m_1 + m_2)^2} \right]$$

$$\boxed{\text{Loss in } K \cdot E = \frac{m_1 m_2 (u_1 - u_2)^2}{2(m_1 + m_2)}}$$

This formula can be used to calculate loss in $K \cdot E$ in collision of two paticles only.

**Example:** Two particles A and B of mass $m$ and one particle C of mass M are kept on the $x$ axis in the order ABC. Particle A is given a velocity $v\hat{i}$. Consequently there are two collisions, both of which are completely inelastic. If the net energy loss because of these collisions is $\frac{7}{8}$ of the initial energy, the value of M is (ignore frictional losses)

(a) 8 $m$ (b) 6 $m$ (c) 4 m (d) 2 $m$

**Soln.** Let $v'$ be final speed of the joint (three) masses.

From conservation of momentum we get,

$$mv = (2m + M)v'$$

$$v' = \frac{mv}{2m + M}$$

Therefore, Loss in $K \cdot E = \frac{1}{2}mv^2 - \frac{1}{2}(M + 2m)v'^2$

According to question

$$\text{Loss in } K \cdot E = \frac{7}{8} \text{ of initial } K \cdot E$$

$$= \frac{7}{8} \cdot \frac{1}{2} mv^2$$

Therefore, $$\frac{7}{16}mv^2 = \frac{1}{2}mv^2 - \frac{1}{2}(M+2m)\frac{m^2v^2}{(2m+M)^2}$$

$$\frac{7}{16}mv^2 = \frac{1}{2}mv^2\left[1 - \frac{m}{2m+M}\right]$$

$$\therefore \quad \frac{7}{8} = 1 - \frac{m}{2m+M}$$

or $$\frac{m}{2m+M} = \frac{1}{8}$$

or $$M = 6m$$

**Example:** A particle is released form some height after time '*t*' it breaks up into two equal parts. One of which flies horizontally with speed equal to half of the speed of the whole particle just before breaking. What is speed of another part.

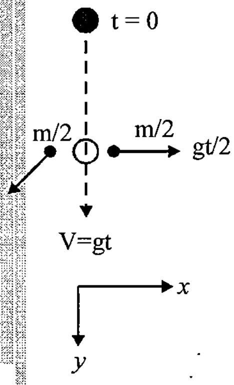
**Soln.** Let $m$ be the mass of particle

Therefore its speed after time '*t*' will be $v = gt$.

Momentum just before and just after breaking must be same.

$$\therefore \quad \vec{p}_{\text{initial}} = \vec{p}_{\text{final}}$$

$$m\,gt\,\hat{j} = \frac{m}{2}\frac{gt}{2}\hat{i} + \frac{m}{2}\vec{V'}$$



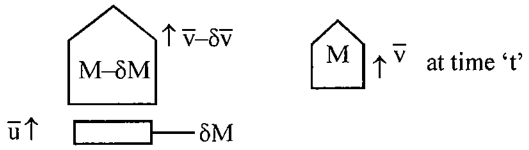
where $\vec{V'}$ is velocity of 2*nd* part

$$\therefore \quad \vec{V'} = 2gt\left(\hat{j} - \frac{\hat{i}}{4}\right)$$

$$\left|\vec{V'}\right| = \frac{\sqrt{17}}{2}gt$$

### *VARIABLE MASS PROBLEM:*

Let a rocket of mass 'M' moving at velocity $\vec{v}$ at any time 't' in earth reference frame. Let at a latter instant $t + \delta t$. $\delta M$ mass of ejected gas jet be moving with velocity $\vec{u}$ and the rocket of mass $M - \delta M$ with new velocity $\vec{v} + \delta\vec{v}$.



Initial momentum $= M\vec{v}$

Final momemtum $= (M - \delta M)(\vec{v} + \delta\vec{v}) + \delta M\vec{u}$

$$F_{\text{ext}} = \frac{\vec{P}_f - \vec{P}_i}{\delta t} = \frac{(M - \partial M)(\vec{v} + \delta\vec{v}) + \delta M\vec{u} - M\vec{V}}{\delta t}$$

$$= M\frac{\delta\vec{v}}{\delta t} - \frac{\vec{v}\delta M}{\delta t} - \delta\vec{v}\frac{\delta M}{\delta t} + \vec{u}\frac{\delta M}{\delta t}$$

Now, as $\delta t \to 0$, $\dfrac{\delta M}{\delta t} = -\dfrac{\delta M}{\delta t}, \dfrac{\delta v}{\delta t} = \dfrac{\delta v}{\delta t}; \delta v \to 0$

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} + V\frac{dM}{dt} - \vec{u}\frac{dM}{dt}$$

$$M\frac{d\vec{v}}{dt} = F_{ext} + (\vec{u} - \vec{v})\frac{dM}{dt} = F_{ext} + V_{rel}\frac{dM}{dt}$$

$V_{rel} = \vec{u} - \vec{v}$ = Relative velocity of exhaust gas w.r.t. rocket.

$F_{ext}$ is the 'external force' acting on the system and $V_{rel}\dfrac{dM}{dt}$ is the 'thrust'.

On gravitational field, $F_{ext} = -Mg$

$$M\frac{d\vec{v}}{dt} = -M\vec{g} + V_{rel}\frac{dM}{dt} = -Mg - V_{rel}\frac{dM}{dt}$$

As M is decreasing, $\dfrac{dM}{dt} = -\text{ive}$

Integrating, $\displaystyle\int_0^V dU = -g\int_0^t dt - V_{rel}\int_{M_0}^{M}\frac{dM}{dt}$

$$V = V_{rel}\log_e\left(\frac{M_0}{M}\right) - gt$$

**Example:** Find the velocity of the rocket assuming that gas is ejected at constant rate $\propto$ and a constant velocity with respect to it and that it moves vertically upwards in a constant gravitational field. Also calculate the total distance travelled by the rocket.

**Soln.** Gas ejected at constant rate $\alpha > 0$; then $m = m_0 - \alpha t$

$$F_{ext} = -mg, \quad \frac{dm}{dt} = -\alpha$$

General equation for rocket is

$$m\frac{dv}{dt} = F_{ext} + v_{rel}\frac{dM}{dt} = -mg + (-v_0)(-\alpha) = -mg + v_0\alpha$$

Or, $$(m_0 - \alpha t)\frac{dv}{dt} = -(m_0 - \alpha t)g + \alpha v_0$$

Or, $$\frac{dv}{dt} = -g + \frac{\alpha v_0}{m_0 - \alpha t}$$

$$V = -gt - V_0\ell n(m_0 - \alpha t) + C_1$$

At $t = 0$, $v = 0$; $\Rightarrow$ $0 = 0 - V_0\ell n m_0 + C_1$ $\Rightarrow$ $C_1 = V_0\ell n\, m_0$

Hence, $V = -gt + V_0\ell n\left(\dfrac{m_0}{m_0 - \alpha t}\right)$

$$\frac{ds}{dt} = -gt + V_0\ell n\left(\frac{m_0}{m_0 - \alpha t}\right)$$

And $$-\frac{gt^2}{2}+V_0\int\left(\ell n\, m_0-\ell n\left(m_0-\alpha t\right)\right)dt$$

$$-\frac{gt^2}{2}+V_0t\ell nm_0+\frac{V_0}{\alpha}\left[\left(m_0-\alpha t\right)\ell n\left(m_0-\alpha t\right)-\left(m_0-\alpha t\right)\right]+C_2$$

$$\left(\int \ell n\ x\ dx = x\ \ell n\ x - x\right)$$

At t = 0, s = 0

$$0=0+0+\frac{v_0}{\alpha}\left(m_0\ell n\ m_0-m_0\right)+C_2$$

$$C_2=-\frac{v_0}{\alpha}\left(m_0\ell nm_0-m_0\right)$$

Then, $$S=-gt^2+v_0\ell n\ m_0+\frac{v_0}{\alpha}\left[\left(m_0-\alpha t\right)\ell n\left(m_0-\alpha t\right)-\left(m_0-\alpha t\right)-m_0\ell nm_0+m_0\right]$$

Or, $$S=-\frac{gt^2}{2}+v_0\left[\left(\ell n\ m_0\right)t+\left(\frac{m_0-\alpha t}{\alpha}\right)\ell n\left(m_0-\alpha t\right)+t-\frac{m_0}{\alpha}\ell nm_0\right]$$

$$=-\frac{gt^2}{2}+v_0\left[t+\left(\frac{m_0-\alpha t}{\alpha}\right)\ell n\left(m_0-\alpha t\right)-\left(\frac{m_0}{\alpha}-t\right)\ell nm_0\right]$$

$$=-\frac{gt^2}{2}+v_0\left[t+\left(\frac{m_0-\alpha t}{\alpha}\right)\ell n\left(\frac{m_0-\alpha t}{m_0}\right)\right]$$

**Center of mass:**

The point where whole mass an object or system of particle can be assumed to be concentrated is called center of mass. Concept of center of mass comes from Newton's equation of motion when dynamics of a system of particles subjected a net external force is collectively discussed.

Consider a system of N particles of masses $m_1, m_2 .......m_N$. Let $\vec{r}_1, \vec{r}_2 .......\vec{r}_N$ be instantaneous position vectors of particles and $\vec{F}_1^e, \vec{F}_2^e ........\vec{F}_N^e$ be external forces on $1^{st}, 2^{nd} .........N^{th}$ particles respectively. Then equation of motion of each particle can be written as

$$\vec{F}_1^e+\sum_{i+1}^{N}\vec{F}_{1j}=m_1\frac{d^2\vec{r}_1}{dt^2}$$

$$\vec{F}_2^e+\sum_{j\neq 2}^{N}\vec{F}_{2j}=m_2\frac{d^2\vec{r}_2}{dt^2}$$

$$\vec{F}_N^e+\sum_{i=1}^{N-1}\vec{F}_{Nj}=m_N\frac{d^2\vec{r}_N}{dt^2}$$

Here $\vec{F}_{ij}$ is internal force between $i^{th}$ and $i^{th}$ particle. Since according to Newton's third law sum of interval forces is zero therefore on adding the above equations we get

$$\sum_{i=1}^{N}\vec{F}_i^e=\frac{d^2}{dt^2}\sum_{i=1}^{N}m_i\vec{r}_i$$

or $$\sum_{i=1}^{N} \vec{F}_i^e = \sum_{i=1}^{N} m_i \frac{d^2}{dt^2}\left(\frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}\right)$$

$$\vec{F}_{total}^e = M_{total}\frac{d^2}{dt^2}\vec{R}_{total}, \text{ where } \vec{R}_{total} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

The above equation is a single equation describing the dynamics if all partcles. It is Newton's equation of motion for whole mass. Therefore $\vec{R}_{total}$ is posotion vector of a point in space where whole mass is assumed to be concentrated. We call it centre of mass. Therefore we write $\vec{R}_{com}$ in place of $\vec{R}_{total}$

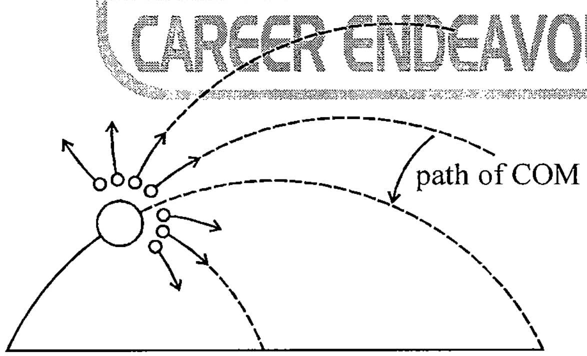
$$\vec{F}_{total}^e = M_{total}\frac{d^2\vec{R}_{com}}{dt^2}$$

or $$\boxed{\vec{F}_{total}^e = M_{total}\vec{a}_{com}}$$

Thus for a system of particles, we can directly apply Newton's equation of motion to center of mass (COM)

Thus we note the following points about dynamics of COM.

- If net external force is zero then center of mass moves with constant velocity.
- If net external force is zero and center of mass is initially at rest then it will always remain at rest although particles of system may start moving due to their mutual internal forces.
- If an object is moving and breaks due to internal forces then center of mass follows the same path as the unbroken object would have followed. For example: If a projectile breaks into two or more particles then also path followed by COM of all particles is still parabolic although various particles may follow different paths.



**Position of center of mass:**

| Discrete | Continuous |
|---|---|
| $\vec{R}_{com} = \dfrac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$ | $\vec{R}_{com} = \dfrac{\int \vec{r}dm}{\int dm}$ |

Coordinates of center of mass:

$$X_{com} = \frac{\sum m_i x_i}{\sum m_i} \qquad X_{com} = \frac{\int x dm}{\int dm}$$

$$Y_{com} = \frac{\sum m_i y_i}{\sum m_i} \qquad Y_{com} = \frac{\int y dm}{\int dm}$$

$$Z_{com} = \frac{\sum m_i z_i}{\sum m_i} \qquad Z_{com} = \frac{\int z dm}{\int dm}$$

**Note:** Center of mass lies at point of symmetry or at line of symmetry.
Velocity of center of mass

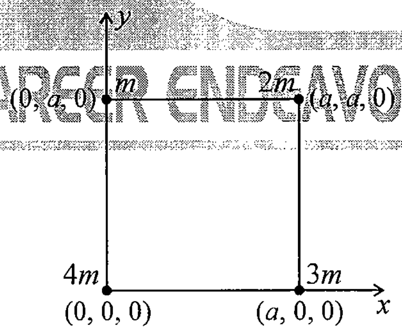
$$\vec{v}_{com} = \frac{\sum_i m_i \vec{v}_i}{\sum m_i}$$

Acceleration of center of mass

$$a_{com} = \frac{\sum_i m_i \vec{a}_i}{\sum m_i} \text{ or } \vec{a}_{com} = \frac{\vec{F}^e_{\text{total}}}{M_{\text{total}}}$$

Above formulae for velocity and acceleration of COM are used when velocity and acceleration of consituents of system are known.

**Example:** Four particles of masses $m$, $2m$, $3m$ and $4m$ are placed at the corners of a square of side '$a$' as shown in figure. If coordinate system is taken as shown $y$. Find position of center of mass.



**Soln.** Coordinates of the masses $m, 2m, 3m$ and $4m$ are $(0, a, 0)$, $(a, a, a)$, $(a, 0, 0)$ and $(0, 0, 0)$ respectively.

Therefore,
$$X_{com} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m \times 0 + 2m \times a + 3m \times a + 4m \times 0}{m + 2m + 3m + 4m} = \frac{a}{2}$$

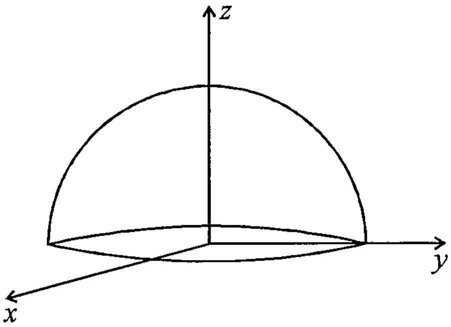
$$Y_{com} = \frac{\sum m_i y_i}{\sum m_i} = \frac{m \times a + 2m \times a + 3m \times 0 + 4m \times 0}{10m} = \frac{3}{10}a$$

Therefore position of center of mass is $\left(\frac{a}{2}, \frac{3a}{10} 10\right)$

**Example:** Find coordinate of center of mass of a thin uniform hollow hemisphere of radius $R$.

**Soln.** Let M be the mass of hollow hemisphere. Let us take coordinate system as shown in figure. $Z$ axis. symmetric therefore center of mass must lie on $Z$ axis.

$\therefore \quad X_{com}=0$ and $Y_{com}=0$



$$Z_{com}=\frac{\int zdm}{\int dm}=\frac{\int zdm}{M}$$

Here $dm$ is mass of small element on surface of sphere

$$\therefore \quad dm=\frac{M}{A}dA=\frac{M}{2\pi R^2}\cdot R^2\sin\theta\cdot d\theta d\phi$$

$$=\frac{M}{2\pi}\sin\theta d\theta d\phi$$

In spherical polar coordinate $Z=R\cos\theta,\ dA=R^2\sin\theta d\theta d\phi$

$$\therefore \quad Z_{com}=\frac{1}{M}\iint R\cos\theta\frac{M}{2\pi}\sin\theta d\theta d\phi$$

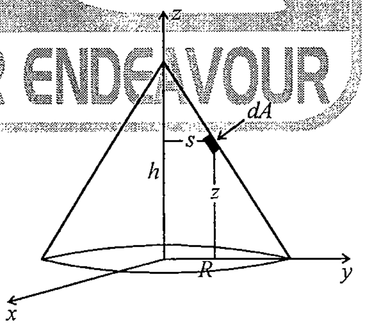
$$=\frac{R}{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\int_0^{2\pi}d\phi\quad=\frac{R}{2\pi}\times\frac{1}{2}\times 2\pi=\frac{R}{2}$$

Therefore center of mass of a thin hollow sphere is $\frac{R}{2}$ distance above its base center.

**Example :** Find position of center of mass of a thin hollow cone of height $h$ and base radius $R$.

**Soln.** Let $M$ be mass of the cone. Let us choose coordinate system as shown in figure. Therefore center of mass must lie somewhere on $Z$ axis.

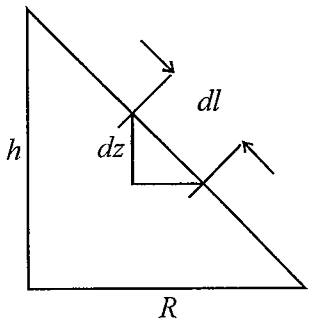
$$\therefore \quad Z_{com}=\frac{\int zdm}{\int dm}=\frac{1}{M}\int zdm$$



Let us use cylindrical coordinates.

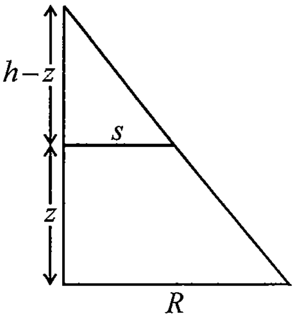
$$\therefore \quad dm=\frac{M}{A}dA=\frac{M}{\pi R\sqrt{R^2+h^2}}\cdot sd\phi\, dl$$

From similarity of small and big triangles shown in figure we get.



$$\frac{dl}{\sqrt{R^2+h}}=\frac{dz}{h} \quad \therefore \quad dl=\frac{\sqrt{R^2+h^2}}{h}\cdot dz$$

Also from similarity of two triangles shown in figure we get,



$$\frac{s}{R}=\frac{h-z}{h}$$

$$s=R\left(1-\frac{z}{h}\right)$$

$$\therefore \qquad dm=\frac{M}{\pi R\sqrt{R^2+h^2}}=\frac{M}{\pi R}\cdot\frac{R\left(1-\frac{z}{h}\right)d\phi}{\sqrt{R^2+h^2}}\cdot\frac{\sqrt{R^2+h^2}}{h}dz$$

or
$$dm=\frac{M}{\pi h}\left(1-\frac{z}{h}\right)dz\,d\phi$$

$$\therefore \qquad Z_{com}=\frac{1}{M}\iint\frac{zM}{\pi h}\left(1-\frac{z}{h}\right)dz\,d\phi$$

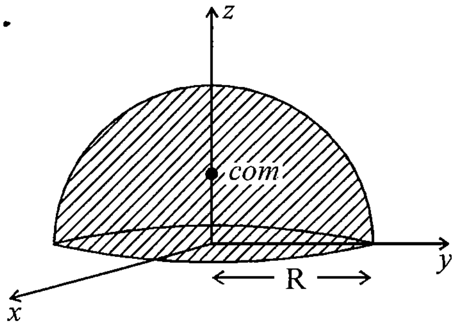
$$=\frac{1}{\pi h}\int_0^h z\left(1-\frac{z}{h}\right)dz\int_0^{2\pi}d\phi$$

$$=\frac{1}{\pi h}\cdot\frac{h^2}{6}\cdot 2\pi=\frac{h}{3}$$

Therefore center of mass of a thin hollow cone of height '$h$' is $\frac{h}{3}$ distance above the base center of cone.

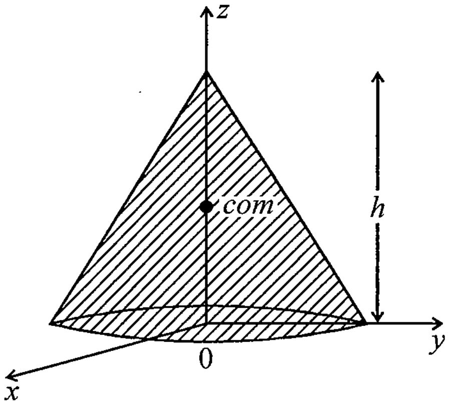
***CENTER OF MASS OF SOME UNIFORM OBJECTS***

- Solid hemisphere



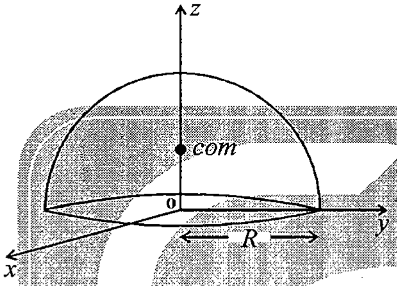
$$Z_{com}=\frac{3R}{8}$$

- Solid cone



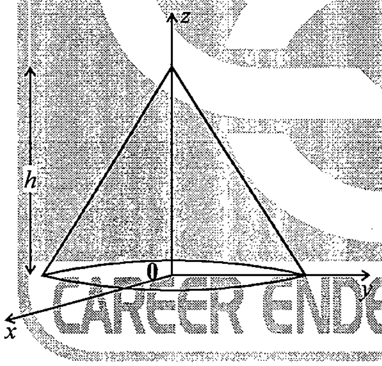
$Z_{com} = \frac{h}{4}$

- Thin hollow hemisphere



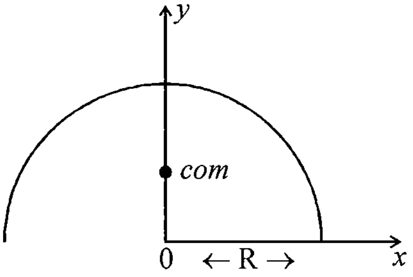
$Z_{com} = \frac{R}{2}$

- Thin hollow cone



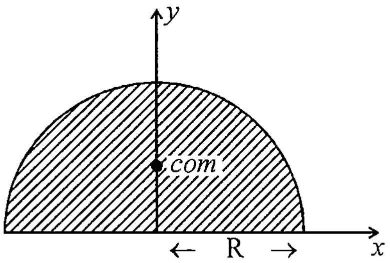
$Z_{com} = \frac{h}{3}$

- Semi ring



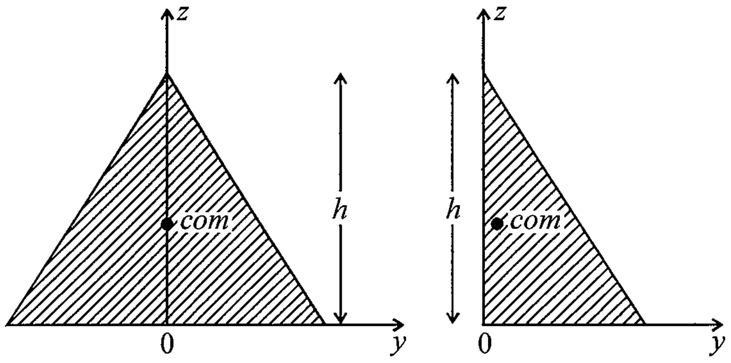
$Y_{com} = \frac{2R}{\pi}$

- Semi disc



$Y_{com} = \frac{4R}{3\pi}$

- Triangle



$Y_{com} = \frac{h}{3}$

## SOLVED PROBLEMS

1. Show that two particles always move in opposite direction in their center of mass frame.

**Soln.** Let $m_1$ and $m_2$ be masses of two particles moving with velocities $\vec{v}_1$ and $\vec{v}_2$.

Therefore, velocity of center of mass is

$$\vec{v}_{com} = \frac{\sum m_i v_i}{\sum m_i} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

velocity of $m_1$ w.r.t. center of mass

$$\vec{v}_{1c} = \vec{v}_1 - \vec{v}_{com} = \vec{v}_1 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}; \quad \vec{v}_{1c} = \frac{m_2(\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

velocity of $m_2$ w.r.t. center of mass

$$\vec{v}_{2c} = \vec{v}_2 - \vec{v}_{com}$$

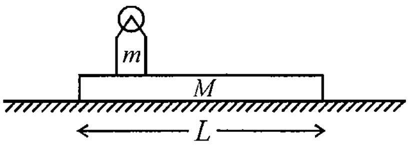
$$= \vec{v}_2 - \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

$$= \frac{m_1(\vec{v}_2 - \vec{v}_1)}{m_1 + m_2} = \frac{m_1(\vec{v}_1 - \vec{v}_2)}{m_1 + m_2}$$

or $\quad \vec{v}_{2c} = -\frac{m_1}{m_2}\vec{v}_{1c}$

Thus the two particles move in opposite direction as seen from center of mass frame.

2. A person of mass $m_1$ stands at one end of a plank of mass $M$ and length $L$. The plank is lying on a smooth horizontal surface. What is distance moves by the plank as the person walks on it to reach the other end.



**Soln.** There is no horizontal force on person + plank system. Therefore center of mass of system will remain stationaly i.e. coordinate of center of mass will not change i.e. $\Delta X_{com} = 0$.

$$\Delta X_{com} = \frac{m_1 \Delta x_1 + m_2 \Delta x_2}{m_1 + m_2} = 0$$

$$\therefore \quad m_1 \Delta x_1 + m_2 \Delta x_2 = 0$$

Here $m_1 = m,\ m_2 = M$

Let plank move rightward by a distance $x$. Therefore distance moved by person is $L + x$.

$$\therefore \quad \Delta x_1 = L + x_1, \quad \Delta x_2 = x$$

$$\therefore \quad m_1(L + x) + Mx = 0$$

$$\therefore \quad x = -\frac{m_1 L}{M + m_1}$$

–ve sign shows that plank moves leftward as person moves rightward. Therefore distance moved by the plank is $\dfrac{mL}{m + L}$

3. A projectile is thrown to get a range $R$. At the highest point it splite into two parts having mass ratio 1:2. The smaller part returns to point of projection. At what distance from point of projection the bigger one lands.

**Soln.** Let $m$ and $2m$ be masses of two fragments. For $m$, $\Delta x_1 = -R$. Since there is no external force in horizontal direction. Therefore,

$$m_1 \Delta x_1 + m_2 \Delta x_2 = 0$$

$$m(-R) + 2m\,\Delta x_2 = 0 \Rightarrow \Delta x_2 = \frac{R}{2}$$

Change in coordinate of bigger part is +ve so it falls furthur away from the point where projectile was expected to land.

So, distance of bigger part from point of projection is $R + \dfrac{R}{2}$ *i.e.* $\dfrac{3R}{2}$

4. A solid sphere of mass $m$ collides head on with identical sphere at rest. If coefficient of restitution is $e$, calculate the loss in kinetic energy.

**Soln.** $e = \dfrac{\text{velocity of separation}}{\text{velocity of approach}}$

Let $u$ be the velocity of a moving sphere before collision and $v_1$ and $v_2$ be the velocities of the two spheres after collision.

$$\therefore \quad e = \frac{v_1 - v_2}{u}$$

$$v_1 - v_2 = eu \qquad \text{... (i)}$$

from conservation of momentum we get

$$mu = mv_1 + mv_2$$

$$\therefore \quad v_1 + v_2 = u \qquad \text{... (ii)}$$

Solving (i) and (ii) we get $v_1 = \dfrac{u(1+e)}{2}$ and $v_2 = \dfrac{u(1-e)}{2}$

Loss of kinetic energy = $\mathrm{K\cdot E}_i - \mathrm{K\cdot E}_f$

$$\Delta K\cdot E = \frac{1}{2}mu^2 - \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2$$

$$= \frac{1}{2}m\left[1 - \frac{(1+e)^2}{4} - \frac{(1-e)^2}{4}\right]u^2$$

$$= \frac{1}{2}mu^2\left[1 - \frac{1+e^2}{2}\right] = \frac{1}{4}mu^2(1-e^2)$$

5. Radius of a spherical water drop increases at the rate $kt$. Assuming its density to be uniform and initial radius $r_0$, calculate the velocity when it has fallen for a time $t$ (neglect air drag and gravity).

**Soln.** $\dfrac{dr}{dt} = kt \qquad \therefore r = r_0 + \dfrac{kt^2}{2}$

Equation of motion $\dfrac{dp}{dt} = F$

In absence of air drag and gravity $F = 0$

$$\therefore \frac{d(mv)}{dt} = 0 \qquad v\frac{dm}{dt} + m\frac{dv}{dt} = 0$$

or $\dfrac{dm}{m} = -\dfrac{dv}{v}$

Let $m_0$ and $v_0$ be respectively initial mass and speed.

Integrating we get $\dfrac{v}{v_0} = \dfrac{m_0}{m}$

or $v = v_0\dfrac{m_0}{m}$

since $m = \dfrac{4}{3}\pi r^3\rho \qquad \because \rho$ = density = constant

$$\therefore \frac{m_0}{m} = \left(\frac{r_0}{r}\right)^3 \qquad \therefore v = v_0\cdot\left(\frac{r_0}{r}\right)^3 = \frac{r_0^3}{\left(r_0 + \dfrac{kt^2}{2}\right)^3}$$

$$v = v_0\left[\frac{1}{\left(1 + \dfrac{kt^2}{2v_0}\right)^3}\right] \qquad \text{or} \quad v = \frac{v_0}{\left(1 + \dfrac{2kt^2}{2v_0}\right)^3}$$

This shows that speed of drop decreases with time and drop comes to rest after a long time. However inreality rain drops attain a constant velocity, because of presence of air drag and gravity.

6. A falling rain drop accumulates moisture due to which its radius increases at a constant rate $k$. Neglecting air drag calculate the speed of rain drop after it has fallen for a time $t$.

**Soln.** Equation of motion $\dfrac{dp}{dt} = mg$ ... (i)

given $\dfrac{dr}{dt} = k \Rightarrow r = kt$ (if initial radius is zero)

since $m = \dfrac{4}{3}\pi r^3 \rho$ $\quad\therefore\quad \dfrac{dm}{dt} = 4\pi r^2 \cdot \dfrac{dr}{dt} \cdot \rho$

$\rho$ is density, we assume it to be constant

$$\frac{dm}{dt} = \left(\frac{4}{3}\pi r^3 \rho\right) \cdot \frac{3k}{r}$$

$$\frac{dm}{dt} = \frac{3mk}{r} \qquad \text{... (ii)}$$

Dividing (i) by (ii) we get

$$\frac{dp}{dm} = \frac{g}{3k} r \qquad dp = \left(\frac{g}{3k}\right) \cdot \left(\frac{3m}{4\pi\rho}\right)^{1/3} dm$$

on integration we get

$$\therefore \quad p = \left(\frac{g}{3k}\right)\left(\frac{3}{4\pi\rho}\right)^{1/3} \cdot \frac{m^{4/3}}{4/3} + c$$

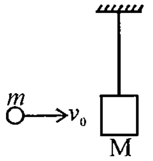
taking $p = m = 0$ at $t = 0$, we get $c = 0$

$$\therefore \quad p = \frac{g}{4k}\left(\frac{3}{4\pi\rho}\right)^{1/3} \cdot m^{4/3}$$

$$\text{or } v = \left(\frac{g}{4k}\right) \cdot \left(\frac{3m}{4\pi\rho}\right)^{1/3}$$

$$v = \frac{g}{4k} r \Rightarrow \boxed{v = \frac{g}{4} t}$$

7. Ballistic pendulum is a device for measuring speed of bullet. Consider a bullet of mass '$m$' striking a ballistic pendulum as shown in the figure. The bullet gets embedded into the block and the system rises to a height $h$. Calculate the initial speed of the bullet.



**Soln.** Let $v_0$ be the initial speed of bullet

Here both momentum and mechanical energy are conserved.

$\therefore$ velocity of bullet + block system just after collision

$$v = \frac{mv_0}{(M+m)}$$

$\therefore$ From conservation of mechanical energy we get (apply after collision)

$$\frac{1}{2}(M+m)v^2 = (M+m)gh$$

or $$\frac{1}{2}(M+m)\cdot\frac{m^2v_0^2}{(M+m)^2} = (M+m)gh$$

$$v_0 = \left(1+\frac{M}{m}\right)\sqrt{2gh}$$

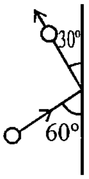
Thus by measuring height, speed of bullet can be determined.

8. A ball hits a wall with velocity 20 m/s at an angle of 60° and rebounds at an angle of 30° with the wall. Calculate the coefficient of restitution.

**Soln.** Since no force acts on the ball along the plane of wall.

Therefore, momentum along the wall remains conserved.

Let $v$ be the velocity with which ball rebounds.



$$\therefore\ m\cdot 20\cos 60 = mv\cos 30$$

$$\therefore\ v = \frac{20}{\sqrt{3}} \text{ m/s}$$

$$\text{coefficient of restitution} = \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

$$= \frac{v\sin 30°}{20\sin 60°} = \frac{\frac{20}{\sqrt{3}}\times\frac{1}{2}}{20\times\frac{\sqrt{3}}{2}} = \frac{1}{3}$$

9. Locate the centre of mass of area between the curves

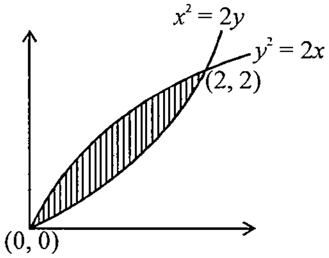
$$y^2 = 2x$$

$$x^2 = 2y$$

**Soln.** Area between the two curves is shown in figure.

Let A be the area between the two curves

$\therefore$ X co-ordinate of centre of mass is

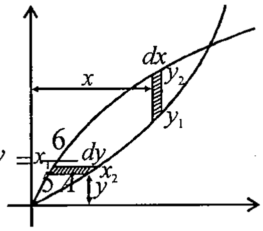


$$X_C = \frac{1}{A}\int_0^2 x\,dA = \frac{1}{A}\int_0^2 x(y_2 - y_1)\,dx$$

$$X_C = \frac{1}{A}\int_0^2 x\left(\sqrt{2x} - \frac{x^2}{2}\right)dx$$

$$= \frac{1}{A}\left[\sqrt{2}\,\frac{x^{5/2}}{5/2} - \frac{1}{2}\cdot\frac{x^4}{4}\right]_0^2$$

$$=\frac{1}{A}\left[\frac{\sqrt{2}\cdot 2^{5/2}}{5/2}-\frac{2^4}{8}\right]=\frac{6}{5A}$$



Y- co-ordinate of centroid is given by

$$Y_C=\frac{1}{A}\int_0^2 ydA=\frac{1}{A}\int_0^2 y(x_2-x_1)dy=\frac{1}{A}\int_0^2 y\left(\sqrt{2}y-\frac{y^2}{2}\right)dy=\frac{6}{5A}$$

Area between the curves

$$A=\int dA=\int_0^2(y_2-y_1)dx=\int_0^2(\sqrt{2}x-\frac{x^2}{2})dx=\left[\sqrt{2}\cdot\frac{x^{3/2}}{3/2}-\frac{1}{2}\cdot\frac{x^3}{3}\right]_0^2=\frac{8}{3}-\frac{4}{3}=\frac{4}{3}$$

$$\therefore\ X_C=\frac{6}{5A}=\frac{6}{5}\times\frac{3}{4}=\frac{9}{10}\qquad Y_C=\frac{6}{5A}=\frac{9}{10}$$

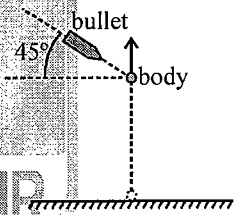
Therefore, coordinate of centroid is $=\left(\frac{9}{10},\frac{9}{10}\right)$

10. A body of mass 9gm is thrown vertically upward with an initial velocity of 100 m/s. After one second, a bullet of mass 1gm hits the body at an angle of 45° with horizontal with a velocity of $90\sqrt{2}$ m/s and sticks to it. Calculate total horizontal distance travelled by the body bullet system. (take $g$ = 10m/s$^2$).

**Soln.** Let the bullet strikes the body as shown in figure, if $v$ be the horizontal component of velocity of body-bullet system after collision then from conservation of horizontal component of linear momentum we get,

$$(1\text{gm})\ (90\sqrt{2}\text{ m/s})\cos 45=10(\text{gm})v$$

$$\therefore\ \text{v}=9\text{m/s}$$



height of the body at the time of collision is

$$h=100\times 1+\frac{1}{2}(-10)\times 1^2\qquad [\text{use } s=ut+\frac{1}{2}at^2]$$

$$h=95\text{ meter}$$

velocity of body just before collision $v=u+at=100-10\times 1=90\text{m/s}$

let $v'$ be vertical component of velocity of body bullet system just after collision.

$$\therefore\ 9\times 90-1\times 9\sqrt{2}\sin 45^\circ=10v'$$

$$890=10v'\Rightarrow v'=89\text{m/s}$$

Time taken by the body-bullet system to reach the ground is obtained using

$$s=ut+\frac{1}{2}at^2,\ -95=89t-\frac{1}{2}\times 10t^2$$

$$5t^2-89t-95=0$$

$$\therefore\ t=\frac{89\pm\sqrt{(89)^2+4\times 5\times 95}}{2\times 5}=18.81\text{ sec}$$

Total horizontal distance moved by bullet-body system is

$$s=vt=9\times 18.81=169.30\text{ meter}$$

11. In an inelastic collision which one of the following is true? **[GP-2006]**

(a) Total energy is not conserved (b) Momentum is not conserved

(c) Kinetic energy is conserved (d) Kinetic energy is not conserved

**Soln.** (d)

12. A mass 0.5kg moving with a speed of 2m/s hits another mass 1kg moving in the same direction with a speed 1m/s, kinetic energy of centre of mass is **[GP-2008]**

(a) 4/3J (b) 3/4J (c) 0J (d) 8/3J

**Soln.** (a) Speed of centre of mass is

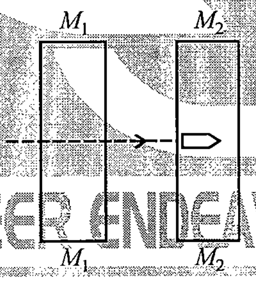
$$v_c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = \frac{0.5\times 2 + 1\times 1}{0.5+1} = \frac{2}{1.5}$$

$$v_c = \frac{2}{1.5} \text{ m/s}$$

kinetic energy of centre of mass

$$= \frac{1}{2}(m_1 + m_2)v_c^2 = \frac{1}{2}(0.5+1)\times\left(\frac{2}{1.5}\right)^2 = \frac{2}{1.5} = 4/3\text{Joule}$$

13. A 20 gm bullet pierces through a plate of mass $M_1 = 1$ kg and then comes to rest inside a second plate of mass $M_2 = 2.98$ kg as shown in figure. It is found that the two plates, initially at rest, now move with equal velocities. Find the percentage loss in the velocity of the bullet when it is between $M_1$ and $M_2$. Neglect any loss of material of the platem due to action of bullet.



**Soln.** Let $v_1$ m/sec be the initial velocity of the bullet and $v_2$ m/sec the velocity with which each plate moves. Here the momentum of the plate $M_2$ and the bullet because the bullet comes to rest inside the second plate $M_2$. Therefore, initial momentum = Final momentum

$$m v_1 = M_1 v_2 + (M_2 + m) v_2$$

$$0.02 v_1 = 1\times v_2 + (2.98 + 0.02) v_2$$

$$v_1 = 200 v_2 \qquad \text{... (i)}$$

Let the velocity of the bullet when it comes out of the plate $M_1$ be $v_3$. The momentum of the bullet between plates $M_1$ and $M_2$ is equal to the momentum of plates $M_2$ plus the bullet

$$0.02 v_3 = (0.02 + 2.98) v_2$$

$$v_3 = 150 v_2 \qquad \text{... (ii)}$$

Now % loss $= \frac{v_1 - v_3}{v_1}\times 100 = \frac{200 v_2 - 150 v_2}{200 v_2}\times 100 = \mathbf{25}$ **%**.

14. A projectile of mass 50 kg is shot vertically upwards with an initial velocity of 100 m/sec. After 5 seconds it explodes into two fragments, one of which having mass 20 kg travels vertically up with a velocity of 150 m/sec.
 (i) What is the velocity of the other fragment at that instant ?
 (ii) Calculate the sum of the momenta of the two fragments 3 second after the explosion. What would have been the momentum of the projectile at this instant if there had been no explosion ?

**Soln.** (i) After explosion, one fragment of mass 20 kg goes upward with velocity 150 m/sec. It is quite obvious that the second fragment of mass 30 kg will go downward with a velocity, say $v$.

The velocity of the projectile after 5 seconds

$$= u - gt$$

$$= 100 - 9.8 \times 5 = 51 \text{ m/sec.}$$

The momentum of projectile before explosion

$$= 50 \times 51 = 2550 \text{ kg m/sec.}$$

Momentum of projectile after explosion

$$= 20 \times 150 - 30v$$

$$2550 = 20 \times 150 - 30v \text{ or } v = \frac{3000 - 2550}{30} = \mathbf{15\ m/sec.}$$

(ii) After 3 seconds velocity of first fragment

$$= 150 - 9.8 \times 3 = 120.6 \text{ m/sec.}$$

Velocity of second fragment after 3 seconds

$$= 15 + 9.8 \times 3 = 44.4 \text{ m/sec.}$$

Sum of the momenta = Momentum of first fragment + Momentum of second fragment

$$= 20 + 120.6 + 30 \times 44.4$$

$$= 2412 + 1332 = \mathbf{3744\ kg\ m/sec}$$

Velocity of projectile after 8 second

$$= 100 - 9.8 \times 8$$

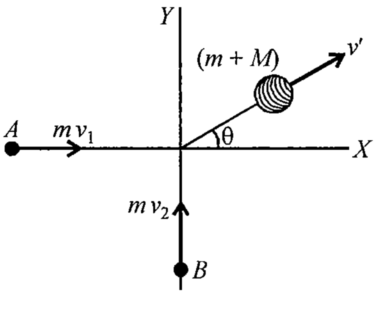
$$= 21.6 \text{ m/sec}$$

Momentum of projectile after 8 sec.

$$= 50 \times 21.6 = \mathbf{1080\ kg\ m/sec.}$$

15. A body of mass $m$ moving with a velocity $v_1$ in the X-direction collides with another body of mass $M$ moving in Y-direction with a velocity $v_2$. They coalesce into one body during collision. Calculate
 (i) the direction and magnitude of the momentum of the final body,
 (ii) the fraction of the initial kinetic energy transformed into heat during the collision in terms of the two masses.

**Soln.** The situation is shown in figure.



(i) Equating the total initial and final momentum along each axis, we get

$$m v_1 + 0 = (M + m) v' \cos\theta \quad \text{... (i)}$$

$$0 + M v_2 = (M+m) v' \sin\theta \qquad \text{... (ii)}$$

Squaring and adding equations (i) and (ii), we get

$$(m v_1)^2 + (M v_2)^2 = (M+m)^2 v'^2 \qquad \text{... (iii)}$$

The final momentum $P$,

$$= (M+m) v' = \sqrt{[(m v_1)^2 + (M v_2)^2]} \qquad \text{from equation (iii)}$$

Dividing equation (ii) by equation (i), we have

$$\tan\theta = \frac{M v_2}{m v_1} \text{ or } \theta = \tan^{-1}\left(\frac{M v_2}{m v_1}\right)$$

(ii) The initial kinetic energy $= \frac{1}{2} m v_1^2 + \frac{1}{2} M v_2^2$

Final kinetic energy $= \frac{1}{2}(M+m) v'^2 = \frac{1}{2} \cdot \frac{1}{(M+m)} \cdot [(m v_1)^2 + (M v_2)^2]$

Decrease in total kinetic energy is given by

$$\Delta E = \left(\frac{1}{2} m v_1^2 + \frac{1}{2} M v_2^2\right) - \frac{1}{2} \cdot \frac{1}{(M+m)} [(m v_1)^2 + (M v_2)^2]$$

$$= \frac{1}{2} \frac{M m}{(M+m)} (v_1^2 + v_2^2)$$

Fraction of kinetic energy transformed into heat

$$= \frac{\Delta E}{\frac{1}{2} m v_1^2 + \frac{1}{2} M v_2^2} = \left(\frac{M m}{M+m}\right)\left(\frac{v_1^2 + v_2^2}{m v_1^2 + M v_2^2}\right).$$

16. A horizontal spring-block-system of mass $M$ executes simple harmonic motion. When the block is passing through its equilibrium position, an object of mass $m$ is put on it and the two move together. Find the new amplitude and frequency of vibration.

**Soln.** Let $A$ and $\omega$ be the initial amplitude and angular frequency of the spring-block-system. Then

$$\omega = \sqrt{\left(\frac{K}{M}\right)},$$

where $K$ = spring constant.

When an object of mass $m$ is placed on the spring, then the new angular frequency $\omega'$ is given by

$$\omega' = \sqrt{\left(\frac{K}{M+m}\right)} = \omega = \sqrt{\left(\frac{M}{M+m}\right)} \qquad \text{... (i)}$$

Let $v$ and $v'$ be the velocities of the system before and after putting the object, then

$$M v = (M+m) v' \qquad \text{... (ii)}$$

$$\frac{1}{2} M v^2 = \frac{1}{2} K A^2 \qquad \text{... (iii)}$$

$$\frac{1}{2}(M+m)(v')^2 = \frac{1}{2} K A'^2 \qquad \text{... (iv)}$$

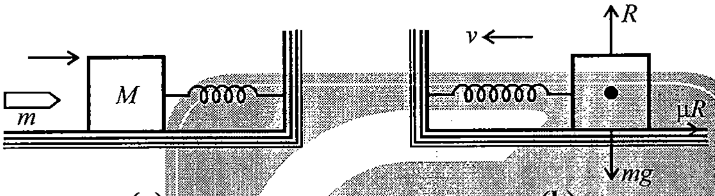
From eq. (ii) $\dfrac{v}{v'}=\left(\dfrac{M+m}{M}\right)$ ...(v)

Dividing eq. (iii) by eq. (iv), we get

$$\frac{M}{(m+M)}\cdot\frac{v^2}{v'^2}=\frac{A^2}{A'^2} \text{ or } \frac{A}{A'}=\frac{v'}{v}\left[\frac{(M+m)}{M}\right]^{1/2}$$

$$\therefore \quad A'=A\left(\frac{v'}{v}\right)\left[\frac{(M+m)}{M}\right]^{1/2}=A\left(\frac{M}{M+m}\right)\left[\frac{(M+m)}{M}\right]^{1/2} \text{ or } A'=A\left[\frac{M}{(M+m)}\right]^{1/2}.$$

17. (a) A rifle bullet of mass 0.1 kg strikes and embeds itself in a block of mass 0.9 kg. The block rests on a horizontal frictionless table figure (a) and is attached to a spring. The impact compresses the spring by 0.2 m. If a force of 1 N compresses the spring by 2 cm, calculate the initial velocity of the bullet. What is the velocity of the block just after the impact ?



(a) (b)

(b) A 2 kg block collides with a horizontal weightless spring of force constant 1.0 nt/m figure (b). The block compresses the spring 6 m from the rest position. Assuming the coefficient of kinetic friction between the block and horizontal surface as 0.25, what was the speed of the block at the instant of collision ?

**Soln.** (a) We know that the elastic potential energy of the spring $=\dfrac{1}{2}kx^2$, where $k$ is the spring constant and x is the compression.

Here, $x=0.2$ m and $k=\dfrac{1}{0.02}=50$ N/m.

Therefore, potential energy of spring $=\dfrac{1}{2}kx^2=\dfrac{1}{2}\times 50\times(0.2)^2=1$ joule.

Let $u$ be the velocity of the bullet before collision and $v$ the velocity of block and bullet after collision. Applying the law of conservation of momentum, we have $mu=(m+M)v$.

Therefore, $v=\dfrac{mu}{m+M}=\dfrac{0.1u}{(0.1+0.9)}=0.1u$ ...(i)

After collision, K.E. of block and bullet = P.E. of the spring.

Therefore, $\dfrac{1}{2}(m+M)v^2=1$ or $v^2=2$ m/sec or $v=\sqrt{2}$ **m/sec.** ...(ii)

From eq. (i), $\sqrt{2}=0.1u$ or $u=\mathbf{10\sqrt{2}}$ **m/sec.**

(b) When the spring is compressed the workdone on the spring is stored as elastic potential energy

Therefore, P.E. of the spring $=\dfrac{1}{2}kx^2$, where $k$ is force constant and $x$ is the compression.

Hence, P.E. $=\dfrac{1}{2}\times 1.0\times(6)^2=18$ joule.

Workdone against frictional force $= \mu \cdot mg \cdot x$

$$= 0.25 \times 2 \times 9.8 \times 6 = 29.4 \text{ joule.}$$

Let $v$ be the initial velocity of the block, then $\frac{1}{2}mv^2 = 18 + 29.4 = 47.4$.

Therefore, $v^2 = \frac{47.4 \times 2}{m} = \frac{47.4 \times 2}{2} = 47.4$ or $v = \mathbf{6.89\ m/sec.}$

18. A moving particle of mass $m$ collides head-on with a particle of mass $2m$ which is initially at rest. Show that the particle $m$ will loose 8/9th part of its initial kinetic energy after the collision.

**Soln.** Let $u_1$ be the initial velocity of mass $m$ before collision and $v_1$ and $v_2$ the velocities of masses $m$ and $2m$ after collision respectively.

According to the law of conservation of kinetic energy, we have

$$\frac{1}{2}m u_1^2 = \frac{1}{2}m v_1^2 + \frac{1}{2}(2m) v_2^2$$

or $\quad u_1^2 - v_1^2 = 2v_2^2$

or $\quad (u_1 - v_1)(u_1 + v_1) = 2v_2^2 \qquad \text{...(i)}$

By the law of conservation of momentum, we have

$$m u_1 = m v_1 + (2m) v_2$$

$$(u_1 - v_1) = 2v_2 \qquad \text{...(ii)}$$

From eqs. (i) and (ii), we get $(u_1 + v_1) = v_2 \qquad \text{...(iii)}$

Substituting the value of $v_2$ from eq. (iii) in eq. (ii), we get

$$u_1 - v_1 = 2(u_1 + v_1) \text{ or } -3v_1 = u_1 \text{ or } v_1 = -\left(\frac{1}{3}\right)u_1 \qquad \text{...(iv)}$$

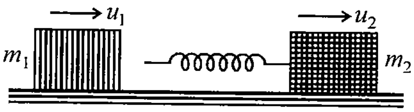
Now the initial and final kinetic energies of mass $m$ are

$$K_i = \left(\frac{1}{2}\right)m u_1^2 \text{ and } K_f = \left(\frac{1}{2}\right)m v_1^2 = \left(\frac{1}{2}\right)m\left(\frac{u_1^2}{9}\right)$$

Therefore, fraction loss $= \frac{K_i - K_f}{K_i} = \frac{\frac{1}{2}m u_1^2\left(1 - \frac{1}{9}\right)}{\frac{1}{2}m u_1^2} = \mathbf{\frac{8}{9}}$.

19. Two blocks of masses $m_1 = 2$ kg and $m_2 = 5$ kg are moving in the same direction along a frictionless surface with speeds 10 m/s and 3 m/s respectively, $m_2$ being ahead of $m_1$. An ideal spring with $k = 1120$ nt/m is attached to the back side of $m_2$. Find the maximum compression of the spring when the blocks collide.

**Soln.** The situation is shown in figure.



Let $v$ be the speed of the system after collision. Applying the law of conservation of energy, we have

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2)v$$

Substituting the given values

$$(2\times10)+(5\times3)=(2+5)v$$

$$v=\frac{20+15}{7}=5\text{ m/s.}$$

Now applying the law of conservation of kinetic energy, we get

$$\frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=\frac{1}{2}(m_1+m_2)v^2+\frac{1}{2}kx^2$$

or $$m_1u_1^2+m_2u_2^2=(m_1+m_2)v^2+kx^2$$

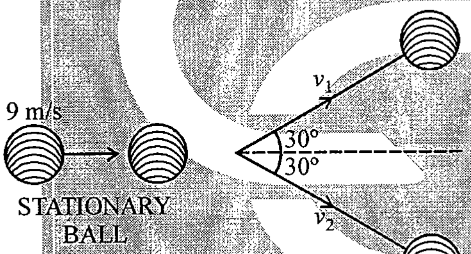
$$(2\times100)+(5\times9)=(7\times25)+(1120\times x^2)$$

$$200+45=175+1120x^2$$

$$x^2=\frac{200+45-175}{1120}=\frac{1}{16}$$

$$x=\frac{1}{4}=\mathbf{0.25\ m.}$$

20. A ball moving with a speed of 9 m/s strikes an identical stationary ball such that after the collision the direction of each ball makes an angle of 30° with the original line of motion figure.



Find the speeds of the two balls after the collision. Is the kinetic energy conserved in this collision process ?

**Soln.** Initial momentum of the balls $= m\times9+m\times0=9m$ ... (i)

where $m$ is the mass of each ball.

Let after collision their velocities are $v_1$ and $v_2$ respectively.

Final momentum of the balls after collision along the same line $= mv_1\cos30+mv_2\cos30$.

$$=\frac{mv_1\sqrt{3}}{2}+\frac{mv_2\sqrt{3}}{2} \qquad \text{... (ii)}$$

According to law of conservation of momentum

$$9m=\frac{mv_1\sqrt{3}}{2}+\frac{mv_2\sqrt{3}}{2}$$

$$\frac{9\times2}{\sqrt{3}}=v_1+v_2 \qquad \text{... (iii)}$$

The initial momentum of the balls along perpendicular direction = 0.

Final momentum of balls along the perpendicular direction $= mv_1\sin30-mv_2\sin30=\frac{m}{2}(v_1-v_2)$.

Again by the law of conservation of momentum $\left(\frac{m}{2}\right)(v_1 - v_2) = 0$.

Therefore, $(v_1 - v_2) = 0$ ... (iv)

Solving equations (iii) and (iv), we have $v_1 = 3\sqrt{3}$ m/s and $v_2 = 3\sqrt{3}$ m/s.

According to law of conservation of energy, Energy before collision = Energy after collision

$$\frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

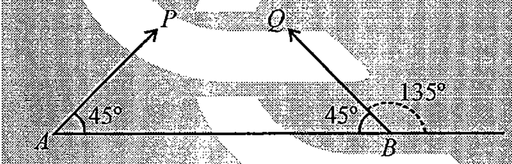
$$\frac{1}{2}m(9)^2 + 0 = \frac{1}{2}m(3\sqrt{3})^2 + \frac{1}{2}m(3\sqrt{3})^2$$

$$\frac{81\,m}{2} = \frac{54\,m}{2}$$

$$L.H.S. \neq R.H.S.$$

i.e., energy is not conserved in this collision or this is a case of inelastic collision.

21. Particles $P$ and $Q$ of mass 20 gm and 40 gm respectively are simultaneously projected from points $A$ and $B$ on the ground. The initial velocities of $P$ and $Q$ make 45° and 135° angles respectively with the horizontal as shown in the figure. Each particle has an initial speed of 49 m/sec. The separation $AB$ is 245 m. Both particles travel in the same vertical plane and undergo a collision. After the collision $P$ retraces its path. Determine the position of $Q$ when it hits the ground. How much time after the collision does the particle $Q$ take to reach the ground. Take $g = 9.8$ m/sec$^2$.



**Soln.** We know that horizontal range $R = u^2 \sin 2\theta / g$.

For particle $P$, $\theta = 45°$, $u = 49$ m/s

$\therefore \quad R = \frac{(49)^2 \sin 90}{9.8} = \frac{(49)^2}{9.8}$ meter

For particle $Q$, $\theta = 180 - 135 = 45°$, $u = 49$ m/s

$\therefore \quad R = \frac{(4.9)^2}{9.8}$ meter

Thus horizontal range is the same for both particles.

As the horizontal range is the same for both the particles, they will meet each other exactly midway between $AB$.

Therefore, horizontal distance $x = u\cos\theta \cdot t$, where $t$ is the time after which the two particles meet.

$$x = \frac{AB}{2} = \frac{245}{2} = 122.5 \text{ meter}$$

$\therefore \quad 122.5 = 49\cos 45° \cdot t$

or $\quad t = \frac{122.5}{49\left(1/\sqrt{2}\right)} = 5/\sqrt{2}$ second ... (i)

At the instant of the collision, the vertical velocities of both particles are zero. The two particles have reached the maximum height. The horizontal velocity is given by

$$v = u\cos\theta = 49\cos 45^\circ = 49/\sqrt{2} \text{ m/s}$$

Initial momentum of two particles

$$= m_1 v - m_2 v = 0.02v - 0.04v = -0.02v \qquad \text{... (ii)}$$

Momenta of two particles after collision $= m_1(-v) + m_2 V$, where $V$ is the velocity of mass $m_2$ after collision.

$$= -0.02v + 0.04V \qquad \text{... (iii)}$$

Therefore, $V = 0$.

The velocity of particle $Q$ becomes zero after collision. Now it falls vertically down midway between $A$ and $B$.

Maximum height $= \dfrac{u^2\sin^2\theta}{2g}$

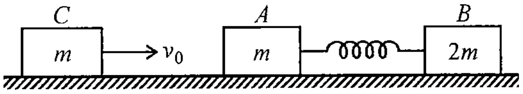
Here $\theta = 45^\circ$, $u = 49$ m/s and $g = 9.8$ m/sec$^2$

Therefore, maximum height $= \dfrac{(49)^2\times\left(\dfrac{1}{\sqrt{2}}\right)^2}{2\times 9.8} = \dfrac{245}{4}$ m

Time of free fall $t$, is given by $\dfrac{1}{2}g t_1^2 = \text{maximum height} = \dfrac{245}{4}$

$$t_1^2 = \frac{245\times 2}{4\times 9.8} \text{ or } t_1 = \frac{5}{\sqrt{2}} \text{ sec.}$$

22. Two bodies $A$ and $B$ of masses $m$ and $2m$ respectively are placed on a smooth floor. They are connected by a spring. A third body $C$ of mass $m$ moves with a velocity $v_0$ along the line joining $A$ and $B$ and collides elastically with $A$, as shown in figure. At a certain instant of time $t_0$ after collision, it is found that the instantaneous velocities of $A$ and $B$ are the same. Further, at this instant the compression of the spring is found to be $x_0$. Determine (i) the common velocity of $A$ and $B$ at time $t_0$; and (ii) the spring constant.



**Soln.** The masses of $C$ and $A$ are same. Initially the body $A$ is not in motion. When the body $C$ collides elastically to body $A$, it stops (masses of two bodies are same) and transfers its momentum $mv_0$ and kinetic energy $(1/2)mv_0^2$ to body $A$. In this position, $B$ is stationary and the spring is in its normal state. Thus, the instantaneous momentum of the system is $mv_0$ and kinetic energy is $(1/2)mv_0^2$. Now the spring compresses and $B$ comes in motion.

Let the instantaneous velocity of $A$ and $B$ after collision be $v$ and the compression of spring $x_0$ (given). As no external force acts on the system, we have, applying the law of conservation of momentum and energy:

$$mv_0 = mv + (2m)v \qquad \text{... (i)}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 + \frac{1}{2}kx_0^2 \qquad \text{... (ii)}$$

From equation (i), $v = \dfrac{1}{3}v_0$ ... (iii)

Substituting the value of $v$ in equation (ii) from equation (iii), we have

$$\frac{1}{2}mv_0^2=\frac{1}{2}m\left(\frac{v_0}{3}\right)^2+\frac{1}{2}(2m)\left(\frac{v_0}{3}\right)^2+\frac{1}{2}kx_0^2$$

Solving we get $k=\dfrac{2mv_0^2}{3x_0^2}$. ... (iv)

23. A ball of mass 100 gm is projected vertically upwards from the ground with a velocity of 49 m/sec. At the same time another identical ball is dropped from a height of 98 m to fall freely along the same path as that followed by the first ball. After some time the two balls collide and stick together and finally fall to the ground. Find the time of flight of the masses.

**Soln.** Let the two balls meet after a time of $t$ second. The distance $s_1$ covered by the ball projected vertically upwards in time $t$ is given by

$$s_1=ut-\frac{1}{2}gt^2=49t-\frac{1}{2}gt^2 \qquad \text{... (i)}$$

The distance $s_2$ covered by the ball dropped from a height 98 m in $t$ second is given by

$$s_2=0+\frac{1}{2}gt^2 \qquad \text{... (ii)}$$

Adding equations (i) and (ii), we get

$$s_1+s_2=49t=98\text{ m} \quad (\because\ s_1+s_2=98\text{ m})$$

$\therefore$ $t=\mathbf{2\ sec.}$

So the two balls collide after 2 seconds.

Before collision, velocity of the ball thrown upwards

$$v_1=49-9.8\times 2=29.4\text{ m/s}$$

Velocity of the ball coming downwards

$$v_2=0+gt=9.8\times 2=19.6\text{ m/s.}$$

Let after collision, $v$ be the velocity of combined mass.

Applying the law of conservation of linear momentum

$$m_1v_1+m_2v_2=(m_1+m_2)v$$

$$-100\times 29.4+100\times 19.6=200v$$

$\therefore$ $v=-4.9$ m/s.

The negative sign shows that the velocity of the combined mass is in the upward direction.

The height to which the combined mass rises after impact

$$=\frac{v^2}{2g}=\frac{(4.9)^2}{2\times 9.8}=1.225.$$

Time taken to cover this distance is given by $v=u-gt$ or $0=4.9-9.8t_1$ or $t_1=\mathbf{0.5\ sec.}$

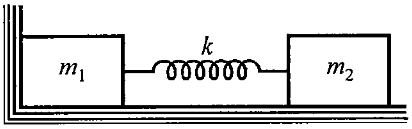
The distance of the combined mass from the ground at this time is given by $s_1+1.225=78.4+1.225=79.625$.

The time $t_2$ taken by the combined mass to reach ground $s=ut+\frac{1}{2}gt^2$ or $79.625=0+\frac{1}{2}\times 9.8\times t_2^2$.

$t_2=\mathbf{4.03\ sec.}$

Therefore, total time of flight of the masses $=t+t_1+t_2=\mathbf{6.53\ sec.}$

24. Two blocks of masses $m_1$ and $m_2$ connected by a weightless spring of stiffness $k$ rest on a smooth horizontal plane. Block 2 is shifted a small distance $x$ to be left and then released. Find the velocity of center of mass of the system after block 1 breaks of the wall.



**Soln.** We know that the potential energy of compression $=\frac{1}{2}kx^2$.

When the block $m_1$ breaks off from the wall, the spring has its unstretched length, and the kinetic energy of block $m_2$ is given by $\frac{1}{2}m_2 v_2^2 = \frac{1}{2}kx^2$.

$$\therefore \quad v_2^2 = \frac{kx^2}{m_2} = x\sqrt{\left(\frac{k}{m_2}\right)}$$

For center of mass $\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

The distances $x_1$ and $x_2$ are measured from the wall.

Again $\frac{d\bar{x}}{dt} = \frac{m_1}{m_1+m_2}\frac{dx_1}{dt} + \frac{m_2}{m_1+m_2}\frac{dx_2}{dt}$

At start $\frac{dx_1}{dt} = 0$

$$\frac{d\bar{x}}{dt} = \frac{m_2 v_2}{m_1+m_2} = \left(\frac{m_2 x}{m_1+m_2}\right)\sqrt{\left(\frac{k}{m_2}\right)} = \frac{x\sqrt{(k\,m_2)}}{m_1+m_2}.$$

# Chapter 5

# Stability Analysis

By stability analysis we mean finding equilibrium positions and investigating whether the given equilibrium is stable or unstable. It is easier to do stability analysis through potential rather than force. Therefore we will try to write potential of given system to discuss equilibrium whenever required.

**Equilibrium criteria in one dimension:**

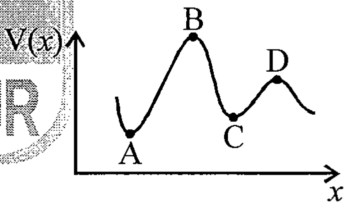
If $V(x)$ be the potential under which a particle is moving then force acting on the particle is $F_x = -\dfrac{dV(x)}{dx}$

at equilibrium point $F_x = 0$. Therefore, condition for equilibrium in terms of potential becomes, $\dfrac{dV(x)}{dx} = 0$

In $V(x)$ versus $x$ graph $\dfrac{dV}{dx} = 0$ at the points where tangent to the curve is parallel to $x$ axis. Therefore in the figure shown below point A, B, C and D are equilibrium points.

**Stable equilibrium point:** A point is stable equilibrium point if, a particle at this point when displaced towards right experiences force towards left and vice versa. That is the force tries to bring it back.

Therefore at stable equilibrium point $\dfrac{dF_x}{dx} < 0$



$$\therefore \frac{d^2V(x)}{dx^2} > 0 \text{ (condition for minimum)}$$

Thus, a stable equilibrium point is a minimum on $V(x)$ versus $x$ plot.
Therefore points A and C in the figure shown above are stable point.

**Unstable equilibrium point:** In this case a particle at equilibrium point is when displaced towards right it experiences force also in rightward direction. That is the force tries to displace the particle away from equilibrium point. Therefore at unstable equilibrium point, $\dfrac{dF_x}{dx} > 0$

$$\therefore \frac{d^2V(x)}{dx^2} < 0 \text{ (condition of maximum)}$$

Thus, an unstable point is a maximum on $V(x)$ versus $x$ plot. Therefore points B and D in the figure shown above are unstable points.

**Frequency of oscillation:** We know a particle of mass $m$ moving under potential $V(x) = V_0 + \frac{1}{2}kx^2$ has frequency of oscillation $\omega = \sqrt{\frac{k}{m}}$, about stable equilibrium point. For a particle moving under some arbitrary potential $V(x)$, if $x_0$ be the stable equilibrium point, then we can expand $V(x)$ about $x_0$ using Taylor's series expansion as,

$$V(x) = V(x_0) + (x - x_0)\left.\frac{dV}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} + \frac{(x-x_0)^3}{\underline{|3}}\left.\frac{d^3V}{dx^3}\right|_{x=x_0} + \ldots\ldots$$

where $(x - x_0)$ is displacement from stable equilibrium point.

At this point, $\left.\frac{dV}{dx}\right|_{x=x_0} = 0$, therefore, if $(x - x_0)$ be small then

$$V(x) = V(x_0) + \frac{(x-x_0)^2}{2}\left.\frac{d^2V}{dx^2}\right|_{x=x_0} \quad \text{(higher power terms neglected)}$$

or $V(x) = V(x_0) + \frac{1}{2}k(x - x_0)^2$

Therefore frequency of oscillation about stable equilibrium is

$$\omega = \sqrt{\frac{k}{m}} \quad \text{where,} \quad k = \left.\frac{d^2V}{dx^2}\right|_{x=x_0}, \; k \text{ is called force constant.}$$

**Equilibrium criteria in two dimension:** If a particle is moving under a two dimensional potential $V(x, y)$ then, at equilibrium point,

$$\left.\frac{\partial}{\partial x}V(x,y)\right|_{x_0,y_0} = 0, \quad \left.\frac{\partial}{\partial y}V(x,y)\right|_{x_0,y_0} = 0$$

for stable equilibrium point (minimum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} > 0, \quad \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} > 0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} \cdot \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} > \left[\left.\frac{\partial^2}{\partial x \partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

For unstable equilibrium point (maximum)

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} < 0, \quad \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} < 0$$

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0} \cdot \left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0} > \left[\left.\frac{\partial^2}{\partial x \partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

for saddle point:

$$\left.\frac{\partial^2}{\partial x^2}V(x,y)\right|_{x_0,y_0}\cdot\left.\frac{\partial^2}{\partial y^2}V(x,y)\right|_{x_0,y_0}<\left[\left.\frac{\partial^2}{\partial x\partial y}V(x,y)\right|_{x_0,y_0}\right]^2$$

## SOLVED PROBLEMS

1. A particle of mass $m$ is moving under a one dimensional potential $V(x)=-ax+bx^2$ where $a>0$ and $b>0$. Find the equilibrium points and find frequency of oscillation about the stable equilibrium.

**Soln.** At equilibrium point, $\left.\frac{dV}{dx}\right|_{x=x_0}=0$

$$\therefore\ -a+2bx_0=0\quad\text{or}\quad x_0=\frac{a}{2b}$$

Thus, $x_0=\frac{a}{2b}$ is an equilibrium point.

To know whether it is stable or unstable equilibrium point let us calculate second derivative of potential at equilibrium point.

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=2b>0$$

Therefore $x_0=\frac{a}{2b}$ is stable equilibrium point, force constant $k=\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=2b$

Therefore, frequency of oscillation is

$$\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{2b}{m}}$$

2. Potential corresponding to force between the atoms of a diatomic molecule is $V(r)=\frac{a}{r^{12}}-\frac{b}{r^6}$ where $a$ and $b$ are positive constants and $r$ is separation between the atoms. Calculate bond length for stable configuration and also calculate frequency of oscillation of atoms if mass of each atom be $m$.

**Soln.** For stable configuration $\left.\frac{dV}{dr}\right|_{r=r_0}=0$

$$-\frac{12a}{r_0^{13}}+\frac{6b}{r_0^7}=0\Rightarrow r_0=\left(\frac{2a}{b}\right)^{1/6}$$

Therefore bond length for stable configuration is $\left(\frac{2a}{b}\right)^{1/6}$

$$\text{Force constant } k=\left.\frac{d^2V}{dr^2}\right|_{r=r_0}=\frac{12\times13a}{r_0^{14}}-\frac{6\times7b}{r_0^8}=\frac{1}{r_0^8}\left(\frac{12\times13a}{r_0^6}-6\times7b\right)$$

$$=\left(\frac{b}{2a}\right)^{8/6}\left(\frac{12\times 13a}{2a/b}-42b\right)=\left(\frac{b}{2a}\right)^{4/3}.36b=\frac{18}{2^{1/3}}.\frac{b^{7/3}}{a^{4/3}}$$

Reduced mass of system $\mu=\frac{m.m}{m+m}=m/2$

Frequency of oscillation $\omega=\sqrt{\frac{k}{\mu}}=\sqrt{\frac{18}{2^{1/3}}.\frac{b^{7/3}}{m/2\,a^{4/3}}}=6\left(\frac{b^7}{2m^3\,a^4}\right)^{1/6}$

3. A particle of mass '$m$' is moving under potential $V(x)=ax^3-bx^2$. Initially the particle is at rest at stable point. What minimum speed be given to the particle so that it reaches unstable point. Plot the potential versus $x$.

**Soln.** For equilibrium point $\left.\frac{dV}{dx}\right|_{x=x_0}=0$

$$3ax_0^2-2bx_0=0\ \Rightarrow x_0=0,\frac{2b}{3a}$$

$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=6ax_0-2b$, $\left.\frac{d^2V}{dx^2}\right|_{x=0}=-2b<0$, $\therefore x_0=0$ is unstable point

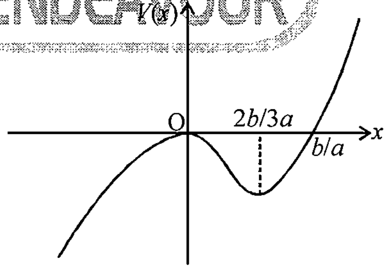
$\left.\frac{d^2V}{dx^2}\right|_{x=\frac{2b}{3a}}=2b>0$, $\therefore\ x_0=\frac{2b}{3a}$ is stable point

To calculate speed let us apply conservation of energy.

Total energy initial = Total energy final

(Kinetic Energy + Potential Energy) at $x=\frac{2b}{a}$ = (Kinetic Energy + Potential Energy) at $x=0$

for minimum speed (u) at stable point the particle will just reach unstable point and stops there.



$$\therefore\ \frac{1}{2}mu^2+V\left(x=\frac{2b}{3a}\right)=\frac{1}{2}m.0^2+V(x=0)$$

$$\frac{1}{2}mu^2+a\left(\frac{2b}{3a}\right)^3-b\left(\frac{2b}{3a}\right)^2=0+0$$

$$u^2=-\frac{2}{m}\left(\frac{2b}{3a}\right)^2\left(\frac{2b}{3}-b\right)=\frac{2}{m}.\frac{4b^2}{9a^2}.\frac{b}{3}=\frac{8b^3}{27ma^2}$$

$$\therefore u=\sqrt{\frac{8b^3}{27ma^2}}$$

4. A particle is moving under potential $V(r)=\frac{a}{r^2}-\frac{b}{r}$. Calculate the minimum value of potential energy.

**Soln.** For potential to be minimum $\left.\frac{dV}{dr}\right|_{r=r_0}=0$

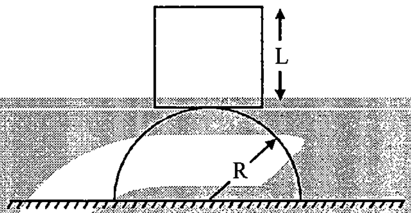
$$-\frac{2a}{r_0^3}+\frac{b}{r_0^2}=0 \qquad \therefore\ r_0=\frac{2a}{b}$$

$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0}=\frac{6a}{r_0^4}-\frac{2b}{r_0^3}=\frac{1}{r_0^3}\left(\frac{6a}{r_0}-2b\right)=\frac{1}{r_0^3}\left(\frac{6a}{2a/b}-2b\right)=\frac{b}{r_0^3}>0$$

Therefore at $r=r_0$ potential is minimum

$$\therefore \qquad V_{\min}=V(r_0)=\frac{1}{r_0}\left(\frac{a}{r_0}-b\right)=-\frac{b^2}{4a}$$

5. A cube is placed on the top of a fixed hemisphere as shown in figure. What should be relation between length of side of cube and radius of hemisphere so that cube has stable equilibrium.



**Soln.** To discuss equilibrium of cube we first write its potential energy as function of angle from its equilibrium position. As shown in the figure below, initially point A was in contact with spherical surface but now in displaced position point B is in contact. Therefore,

$AB=R\theta$

Height of centre of cube from centre level of hemisphere is

$$h=OC+O'D$$

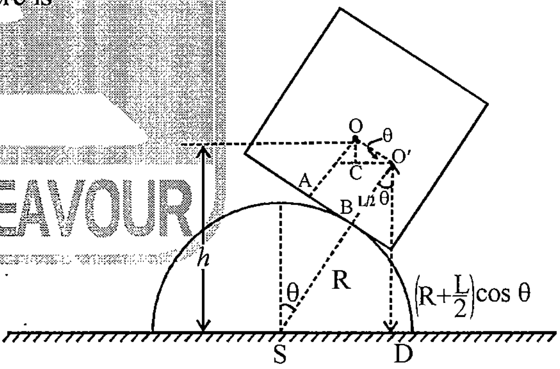
$$=OO'\sin\theta+O'S\cos\theta$$

$$=AB\sin\theta+(O'B+BS)\cos\theta$$

$$=R\theta\sin\theta+\left(\frac{L}{2}+R\right)\cos\theta$$

Potential energy of the cube

$$V(\theta)=Mgh \quad =Mg\left[R\theta\sin\theta+\left(\frac{L}{2}+R\right)\cos\theta\right]$$



$\theta=0$ is equilibrium position of the cube. For this position to be stable equilibrium position, $\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0}>0$

$$\therefore\ \frac{d^2}{d\theta^2}Mg\left[R\theta\sin\theta+\left(\frac{L}{2}+R\right)\cos\theta\right]_{\theta=0}>0$$

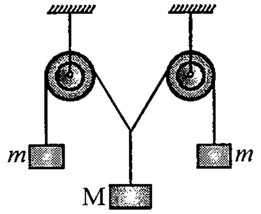
$$\text{or}\ \frac{d}{d\theta}\left[R\theta\cos\theta+R\sin\theta-\left(\frac{L}{2}+R\right)\sin\theta\right]_{\theta=0}>0$$

$$\text{or}\ \left[R\cos\theta-R\theta\sin\theta+R\cos\theta-\left(\frac{L}{2}+R\right)\cos\theta\right]_{\theta=0}>0$$

$$\text{or}\quad \left[2R-\left(\frac{L}{2}+R\right)\right]>0 \quad \therefore R-\frac{L}{2}>0 \quad \text{or} \quad 2R>L$$

Thus, cube can be in stable equilibrium position if its side length is less than diameter of hemisphere.

6. For the mass pulley system shown in figure, what should be relation between $m$ and $M$ so that system remains in stable equilibrium position. Pulley are smooth and strings are tight and inextensible.

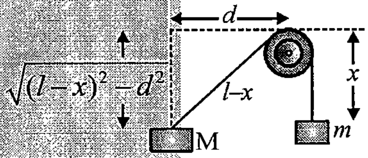


**Soln.** The pulleys are fixed. Therefore we can write potential energy of the system by specifying position of blocks with respect to pulleys.

Let '$l$' be length of string and '$d$' be the half distance between two pulleys. Therefore, $l$ and $d$ are constants. If $x$ be distance of $m$ below the pulley as shown in figure then potential energy of system is

$$V(x)=-2mgx-Mg\sqrt{(l-x)^2-d^2} \quad \therefore \frac{dV}{dx}=-2mg+\frac{Mg(l-x)}{\sqrt{(l-x)^2-d^2}}$$

for equilibrium $\left.\frac{dV}{dx}\right|_{x=x_0}=0$ $\quad \therefore -2mg+\frac{Mg(l-x_0)}{\sqrt{(l-x_0)^2-d^2}}=0$



$$\text{or}\quad \frac{2m}{M}=\frac{l-x_0}{\sqrt{(l-x_0)^2-d^2}} \qquad \text{... (a)}$$

$$\left.\frac{d^2V}{dx^2}\right|_{x=x_0}=\frac{Mgd^2}{\left[(l-x_0)^2-d^2\right]^{3/2}}>0 \text{ for all values of } d>0$$

since, $\frac{l-x_0}{\sqrt{(l-x_0)^2-d^2}}>1$. $\quad \therefore$ from (a) $\frac{2m}{M}>1$ or $2m>M$

7. The potential energy between two atoms are given $v(r)=\frac{a}{r^{12}}-\frac{b}{r^6}$

where $a$, $b$ positive constants.

(i) Find the equilibrium distance of two atoms.

(ii) Plot the potential

(iii) Calculate the frequency of small oscillation.

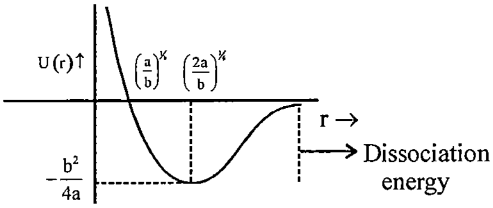
**Soln.** (i) For equilibrium potential energy should be minimum.

$$\frac{dU}{dr}=-\frac{12a}{r^{13}}+\frac{6b}{r^7}=0 \Rightarrow r^6=\frac{2a}{b}$$

$$r=\left(\frac{20}{b}\right)^{1/6}=r_0 \qquad \text{[Position of stability equilibrium]}$$

$$U(r_0)=\frac{ab^2}{4a^2}-\frac{b.b}{2a}=\frac{b^2}{4a}-\frac{b^2}{2a}=-\frac{b^2}{4a}$$

(ii)



(iii)

Approximate to harmonic oscillator

$$U(r_0)=\frac{-b^2}{4a};\ U(r)=U(r_0)+(r-r_0)\left.\frac{dU}{dr}\right|_{r_0}+\frac{(r-r_0)^2}{2}\left.\frac{d^2U}{dr^2}\right|_{r_0}$$

$$\frac{d^2U}{dr^2}=\frac{12\times 13a}{r^{14}}-\frac{42b}{r^8};\ \left.\frac{d^2U}{dr^2}\right|_{r_0}=\frac{156a}{r^{14}}-\frac{42b}{r^8}=\frac{156a}{(2a)^{14/6}}-\frac{42b}{\left(\frac{2a}{h}\right)^{8/6}}$$

$$=\frac{156\,a}{\left(\frac{2a}{b}\right)^{7/3}}-\frac{42b}{\left(\frac{2a}{b}\right)^{4/3}}=\text{constant}=c$$

(Force constant equivalent to spring constant

$$U(r)=U(r_0)+\frac{c}{2}(r-r_0)^2$$

So, the force $=-\frac{dU}{dr}=-\frac{c}{2}.2(r-r_0)^2=-c(r-r_0)$

Force $=-\nabla U;\ U=U(r)$ only

$\vec{F}=m\,\vec{a}$ (Here introduced to harmonic oscillator may necessary).

Equation of motion, $m\frac{d^2r}{dt^2}=-c(r-r_0);\ \frac{d^2r}{dt^2}+\frac{c}{m}(r-r_0)=0$

Take $c=m\omega^2;\ \omega=\sqrt{\frac{c}{m}}$

8. A particle of mass 'm' moving in a potential

$V(x)=\frac{1}{2}m\omega_0^2x^2+\frac{a}{2mx^2}$ $\quad(\omega_0$ & are constant$)$

Find the angular frequency of small oscillation.

$$\frac{dv}{dx}=m\omega_0^2x-\frac{a}{m\,x^3}=0$$

$$m\omega_0^2x_0-\frac{a}{mx^3}=0;\ x_0^4=\frac{a}{m^2\omega_0^2};\ \chi_0=\left(\frac{a}{m^2\omega_0^2}\right)^{1/4}$$

Equilibrium distance, $\frac{d^2z}{dx^2}=m\omega_0^2+\frac{3a}{mx^4}$

$$\left.\frac{d^2r}{dx^2}\right|_{x=x_0} = m\omega_0^2 + \frac{3a}{ma}m^2\omega^2 = 4m\omega_0^2 \qquad \left(x_0^4 = \frac{a}{m^2\omega_0^2}\right)$$

$$U(x) = U(x_0) + (x-x_0)\left.\frac{dU}{dx}\right|_{x=x_0} + \frac{(x-x_0)^2}{2}\left.\frac{d^2U}{dx^2}\right|_{x=x_0}$$

$$= U(x_0) + (x-x_0)^2\, 2m\omega_0^2$$

$$\text{Force} = -\frac{dU}{dx} = -4m\omega_0^2(x-x_0)$$

Equation of motion, $m\frac{d^2x}{dt^2} = F = -4\omega_0^2(x-x_0)$

$$\frac{d^2x}{dt^2} + 4\omega_0^2(x-x_0) = 0$$

Hence, the frequency of small oscillation, $\omega = \sqrt{4\omega_0^2} = 2\omega_0$.

9. A particle of mass '*m*' is constrained to move in one dimension under a potential $u(x) = \frac{\alpha}{x^2} - \frac{\beta}{x}$, where $\alpha$ and $\beta$ are positive constants. Show that period of small oscillations about the equilibrium point is, $T = 4\pi\sqrt{\frac{2\alpha^3 m}{\beta^4}}$.

**Soln.** At equilibrium point $(x = x_0)$

$$\left.\frac{du}{dx}\right|_{x=x_0} = 0$$

$$-\frac{2\alpha}{x_0^3} + \frac{\beta}{x_0^2} = 0 \qquad \therefore\ x_0 = \frac{2\alpha}{\beta}$$

time period of oscillation is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where *k* is force constant and is given by

$$k = \left.\frac{d^2u}{dx^2}\right|_{x=x_0} = \frac{6\alpha}{x_0^4} - \frac{2\beta}{x_0^3} = 6\alpha\cdot\frac{\beta^4}{16\alpha^4} - \frac{2\beta^4}{8\alpha^3}$$

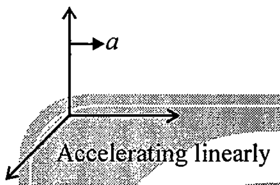
$$= \frac{3}{8}\cdot\frac{\beta^4}{\alpha^3} - \frac{2}{8}\cdot\frac{\beta^4}{\alpha^3} \quad \Rightarrow k = \frac{\beta^4}{8\alpha^3}$$

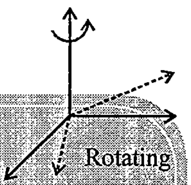
$$\therefore\ T = 2\pi\sqrt{\frac{8\alpha^3}{\beta^4}\cdot m} = 4\pi\sqrt{\frac{2\alpha^3 m}{\beta^4}}$$

# Chapter 6

# *Non-inertial frames and pseudoforces*

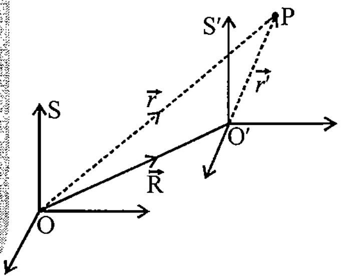
A non-inertial frame is that which is accelerated. It might be accelerating translationally or might be rotating. In the figure shown below two non-inertial fames are shown.





When observation is made from non-inertial frame then it appears that Newton's laws of motion are not being obeyed. The reason is that it appears as if in addition to the known forces some unknown forces are also acting the system. The unknown forces are called pseudoforces or fictitious forces.

**Linearly accelerated frame:** Let us consider a frame $S'$ which is linearly accelerated with acceleration $\vec{a}_0$ with respect to an inertial frame $S$.



From figure, $\vec{R}$ = position vector of $S'$ w.r.t. $S$

$$\therefore\ \vec{a}_0 = \frac{d^2\vec{R}}{dt^2}$$

If $\vec{F}$ be the known forces acting on a particle as measured in $S$ frame

then, $\vec{F} = \dfrac{md^2\vec{r}}{dt^2} = m\vec{a}$, where $\vec{a}$ is acceleration of particle in $S$ frame.

From figure, $\vec{r}' = \vec{r} - \vec{R}$ or $\dfrac{d^2\vec{r}'}{dt^2} = \dfrac{d^2\vec{r}}{dt^2} - \dfrac{d^2\vec{R}}{dt^2}$

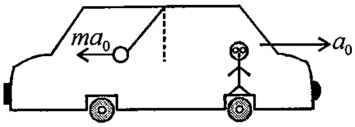
If $\vec{a}'$ be acceleration of particle in $\vec{S}'$ then from above relation we get $\vec{a}' = \vec{a} - \vec{a}_0$

Force on the particle as seen from $S'$

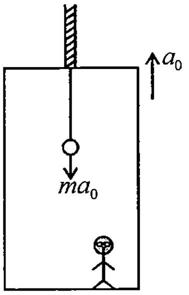
$\vec{F}' = m\vec{a}' = m(\vec{a} - \vec{a}_0)$ $\qquad \therefore\ \vec{F}' = \vec{F} - m\vec{a}_0$

from above relation we see that net force in $S'$ frame consists of a term $-m\vec{a}_0$ in addition to the known forces $\vec{F}$, clearly $-m\vec{a}_0$ is coming into picture because of acceleration of $\vec{S}'$ frame. Therefore in linearly accelerated frame, pseudo force = $-m\vec{a}_0$

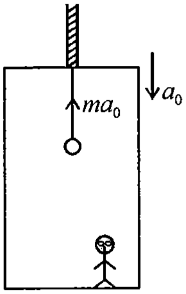
minus sign shows that pseudo force acts opposite to the direction of acceleration of the frame. Pseudo force in some linearly accelerated frames is shown in figure.



Horizontally accelerating car



Upward accelerating lift



Downward accelerating lift

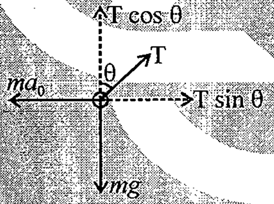
**Note:** The word pseudo force is used by an observer standing on the non-inertial frame. An observer in inertial frame cannot use the word pseudo force. However observer on non-inertial frame finds that pseudo force acts on all objects including the observer itself irrespective of location of the objects. That is objects may or may not be on non-inertial frame.

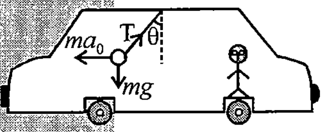
**Example:** A simple pendulum consisting of a point mass $m$ and a light string of length '$l$' is suspended from the roof of a car which is moving horizontally with constant acceleration $a_0$. Find the angle made by string with vertical.

**Soln.** As seen from car the pseudo force $ma_0$ will act on the bob opposite to the direction of acceleration of car due to which string will deflect from its vertically downward position.

Let $\theta$ be the angle made by string with downward vertical in the equilibrium position. The known forces acting on the bob are $mg$ and tension T as shown in figure.

For equilibrium of bob, we must have,



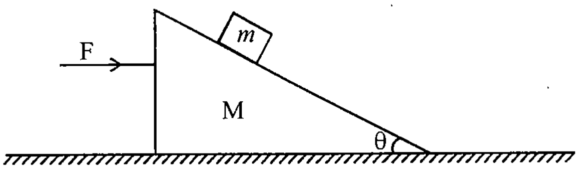


$$T\sin\theta = ma_0, \quad T\cos\theta = mg$$

Dividing the above two relations we get.

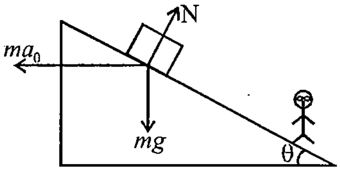
$$\tan\theta = \frac{a_0}{g} \text{ or } \theta = \tan^{-1}\frac{a_0}{g}$$

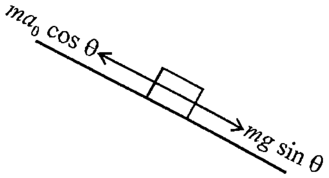
**Example:** A small block of mass '$m$' lies on a wedge of mass M as shown in figure. All the contact surfaces are smooth. When a horizontal force F is applied to the wedge, the block does not slide on the wedge. What must be value of F.



**Soln.** Horizontal acceleration of the system is $a_0 = \dfrac{F}{M+m}$

According to question, the block does not slide on the wedge, therefore if block is seen from the reference frame of the wedge it will appear stationary. The wedge has a linear acceleration $a_0$ therefore if observation is made from the wedge frame a pseudo force $-ma_0$ must be applied on the block as shown in figure. Block is stationary on the wedge so component of forces parallel to the inclined plane must cancel.

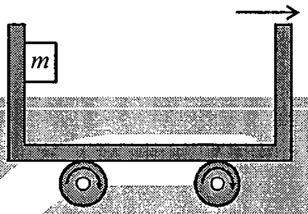




Therefore, $mg\sin\theta = ma_0\cos\theta$ or $\tan\theta = \frac{a_0}{g} = \frac{F}{(M+m)g}$

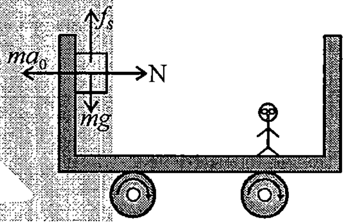
or $F = (M+m)g\tan\theta$

**Example:** A trolley is moving horizontally with a constant acceleration as shown in figure. A block remains stationary on the vertical wall of the trolley. If coefficients of static friction between the wall and block is $\mu$ what must be acceleration of the trolley.



**Soln.** As seen from the trolley's frame the different forces acting on the block are shown in figure. Let $a_0$ be acceleration of trolley. $f_S$ and N are static friction and normal reaction respectively. For equilibrium of block we must have

$f_S = mg$ and $N = ma_0$



We know that $f_S \le \mu N$ $\therefore mg \le \mu m a_0$ $\therefore a_0 \ge \frac{g}{\mu}$

Therefore acceleration of trolley must either be equal to or greater than $g/\mu$.

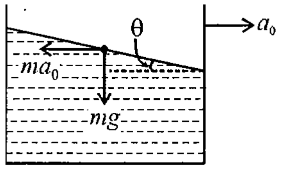
**Example:** A vessel containing water is linearly accelerated due to which surface of water does not remain horizontal. If $a_0$ be acceleration of vessel what is angle made by water surface with horizontal.

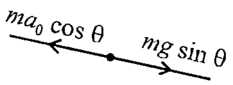
**Soln.** Due to pseudo force water will move opposite to direction of acceleration. Therefore surface of water becomes inclined as shown in figure. Consider a particle of water of mass $m$ right at the surface. If $\theta$ be the angle made by water surface with horizontal then component of pseudo force $-ma_0$ and $mg$ parallel to the surface are as shown in figure.

For equilibrium these forces must cancel.

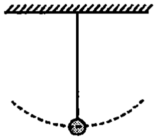
Therefore $ma_0\cos\theta = mg\sin\theta$ or $\tan\theta = \frac{a_0}{g}$

$\theta = \tan^{-1}\frac{a_0}{g}$





**Time period of simple pendulum in linearly accelerated frame:** We know that time period of simple pendulum in an inertial frame when point of suspension is stationary is given by



$$T = 2\pi\sqrt{\frac{l}{g}}$$

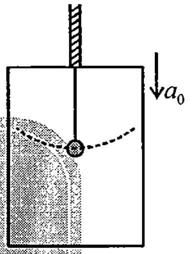
when simple pendulum is suspended in such a way that point of suspension moves with acceleration $\vec{a}_0$ then time period for small oscillation about the equilibrium point is given as

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

where $g_{eff}$ is effective value of acceleration due to gravity and it is given as $g_{eff} = |\vec{g} - \vec{a}_0|$

**Special cases:**

(i) Lift accelerating downward.

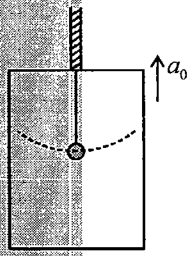


In this case $\vec{a}_0$ is parallel to $\vec{g}$.

Therefore $\therefore g_{eff} = g - a_0$, $a_0 < g$

$$\therefore T = 2\pi\sqrt{\frac{l}{g - a_0}}$$

(ii) Lift accelerating upward



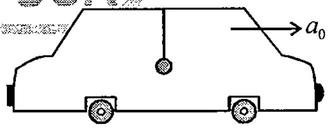
In this case $\vec{a}_0$ is opposite to $\vec{g}$

$$\therefore \; g_{eff} = g + a_0$$

$$\therefore T = 2\pi\sqrt{\frac{l}{g + a_0}}$$

(iii) Car accelerating horizontally.

In this case $\vec{a}_0$ is perpendicular to $\vec{g}$. Therefore

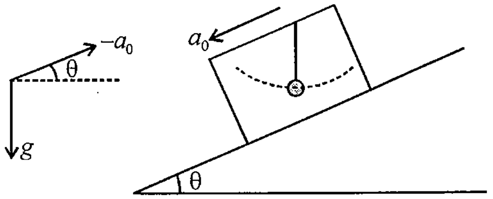


$$g_{eff} = \sqrt{g^2 + a_0^2}$$

$$\therefore T = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + a_0^2}}}$$

**Example:** A large box is placed on a smooth inclined plane of inclination $\theta$ due to which the box slides down. Simple pendulum is suspended from the top of the box inside it. Calculate time period of small oscillation of the pendulum.

**Soln.** Acceleration of box down the inclined plane is $a_0 = g\sin\theta$



Angle between $-a_0$ and $g$ in this case is $90 + \theta$

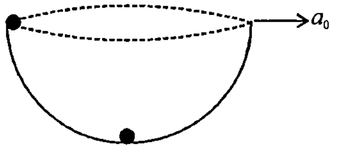
$$g_{eff} = |\vec{g} - \vec{a}_0|$$

$$= \sqrt{g^2 + a_0^2 + 2ga_0\cos(90 + \theta)}$$

$$= \sqrt{g^2 + g^2\sin^2\theta - 2g^2\sin^2\theta} \;=\; g\cos\theta$$

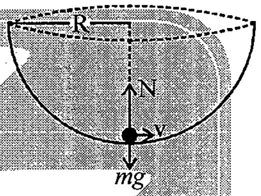
$$\therefore T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g\cos\theta}}$$

**Example:** A small particle of mass $m$ is released at the edge of a smooth bowl as shown in figure. The bowl is accelerated horizontally with constant acceleration $a_0$. The particle slides along the surface of bowl. Calculate normal reaction on the particle when it reaches the bottom.



**Soln.** This problem gets simplified if we observe motion of the particle with respect to the bowl. In bowl's frame particle moves simply in vertical circle of radius equal to radius of bowl. And it is acted upon by $mg$, $-ma_0$ and normal reaction.

Let v be the speed of particle w.r.t. bowl at the lowest point. Therefore



$$N - mg = \frac{mv^2}{R}$$

or $N = mg + \dfrac{mv^2}{R}$ ....(a)

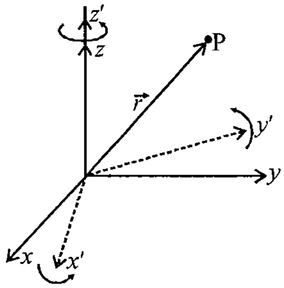
To calculate v let us apply work energy theorem.

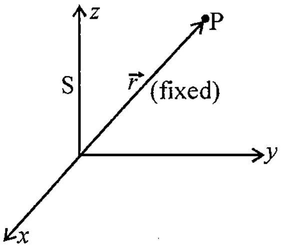
Total work done by all forces = Change in kinetic energy

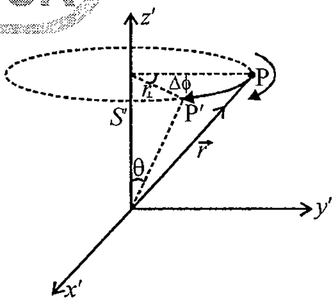
$$mgR - ma_0R = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2(g-a_0)R}$$

from (a) $N = mg + \dfrac{m2(g-a_0)R}{R}$, $N = m(3g-a_0)$

**Rotating Frame:** Let $S'$ frame is rotating with respect to $S$ frame. Therefore a vector which is fixed in $S$ frame will appear to be rotating in $S'$ in a direction opposite to the rotation of $S'$.







For convenience we have assumed that origin and $z$-axes of $S$ and $S'$ coincide. In figures above we have shown the two frames together and a fixed vector as seen from the two frames separately.

As seen from $S'$ a vector which is fixed in $S$ frame describes a cone in $S'$ frame. Therefore change in vector $\vec{r}$ in small time $\Delta t$ as seen frame $S'$ is

$$= p\vec{p}'$$

$$= r_\perp \Delta\phi\hat{e} = r\sin\theta\omega\Delta t\hat{e} \quad \{\hat{e} \text{ is unit vector parallel to } p\vec{p}'\}$$

$$= (r\omega\sin\theta\hat{e})\Delta t = -\vec{\omega}\times\vec{r}\Delta t$$

Thus a fixed vector $\vec{r}$ appears to change by an amount $-\vec{\omega}\times\vec{r}\Delta t$ in time $\Delta t$ when looked at from a rotating frame. However if the vector $\vec{r}$ is not fixed in $S$ frame and if it changes by $\Delta\vec{r}$ in $S$ frame in time $\Delta t$ then change in vector as seen frame $S'$ will include extra change due to rotation also. Therefore in $S'$ frame change in vector will be $\Delta\vec{r}' = \Delta\vec{r} - \vec{\omega}\times\vec{r}\Delta t$

$$\therefore (\Delta\vec{r})_{\text{in } S'} = (\Delta\vec{r})_{\text{in } S} - \vec{\omega}\times\vec{r}\Delta t$$

or $$\left(\frac{\Delta\vec{r}}{\Delta t}\right)_{\text{in } S'} = \left(\frac{\Delta\vec{r}}{\Delta t}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$$

In the limit $\Delta t \to 0$

$$\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$$

The same concept can be applied to all vectors because all vectors will change in similar way when observed from rotating frame. For example when applied to angular momentum we get

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{L}}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{L}$$

**Pseudo forces in rotating frame:** Let us consider dynamics of a particle of mass $m$ in $S$ and $S'$ frames. Let $\vec{r}(t)$ be instantaneous position vector of particle. Therefore in S frame (inertial frame)

Velocity of particle $\vec{v} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S}$

Acceleration of particle $\vec{a} = \left(\frac{d\vec{v}}{dt}\right)_{\text{in } S}$

Net force on the particle $\vec{F} = m\left(\frac{d\vec{v}}{dt}\right)_{\text{in } S}$ where $\vec{F}$ is resultant of all known forces.

In $S'$ frame (rotating frame)

Net force on the particle is $\vec{F}' = m\left(\frac{d\vec{v}'}{dt}\right)_{\text{in } S'}$

From relation $\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S'} = \left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{r}$

We get $\vec{v}' = \vec{v} - \vec{\omega}\times\vec{r}$ and $\left(\frac{dv'}{dt}\right)_{\text{in } S'} = \left(\frac{dv'}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{v}'$

$$\therefore \vec{F}' = m\left[\left(\frac{d\vec{v}'}{dt}\right)_{\text{in } S} - \vec{\omega}\times\vec{v}'\right] = m\left[\frac{d}{dt}(\vec{v} - \vec{\omega}\times\vec{r})_{\text{in } S} - \vec{\omega}\times\vec{v}'\right]$$

$$= m\left(\frac{dv}{dt}\right)_{\text{in } S} - m\left(\frac{d\omega}{dt}\times\vec{r}\right)_{\text{in } S} - m\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\text{in } S} - m\vec{\omega}\times\vec{v}'$$

$$\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt}\times\vec{r} - m\vec{\omega}\times\vec{v} - m\vec{\omega}\times\vec{v}'$$

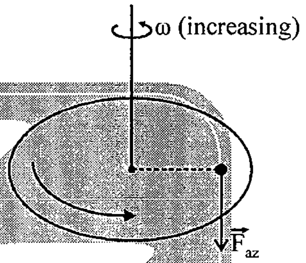
Since observation is being made with respect to the rotating frame therefore we would like to express $\vec{F}'$ in terms of velocity w.r.t. rotating frame i.e. $\vec{v}'$, therefore let us remove $\vec{v}$ by using $\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}$
Therefore

$$\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}) - m\vec{\omega} \times \vec{v}'$$

$$\boxed{\vec{F}' = \vec{F} - m\frac{d\vec{\omega}}{dt} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}'}$$

Thus, as seen from $S'$, in addition to the a known force $\vec{F}$ some unknown forces also appear to be acting on the particle. These forces are called pseudo forces. In the following we discuss each term of pseudo forces one by one.

**Azimuthal Force:** It appears to act on the particles which are being observed from a rotating frame which has non-uniform angular velocity i.e. $\frac{d\omega}{dt} \neq 0$.



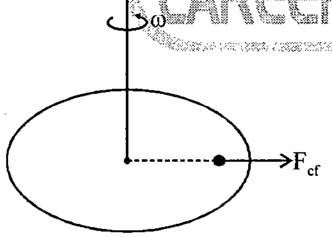
Its direction is tangential to the rotation of frame.

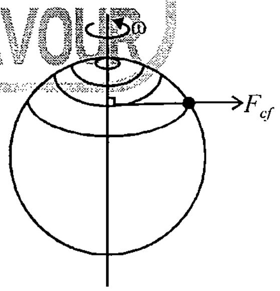
Azimuthal Force $= -m\frac{d\vec{\omega}}{dt} \times \vec{r} = \vec{F}_{az}$

In the figure direction of azimuthal force on a particle lying on a non-uniformly rotating disc is shown.

**Centrifugal Force:** It acts on all particles which are being observed from the rotating frame, irrespective of state of the particle. Direction of centrifugal force is perpendicular to the axes of rotation and away from it.

Centrifugal Force $= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{F}_{cf}$ or $|\vec{F}_{cf}| = m\omega^2 r_\perp$,





where $r_\perp$ is ⊥ distance from axis of rotation

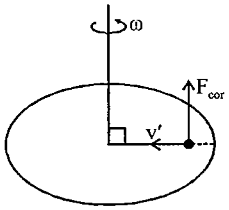
In the figure shown direction of centrifugal force on a particle lying on a disc and one lying surface of a rotating sphere is shown.

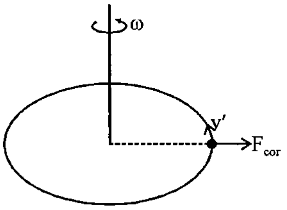
**Coriolis Force:** It acts on all particles which are being observed from the rotating frame provided that particles have non-zero velocity with respect to the rotating frame. That is, coriolis force acts only on those particles which are moving with respect to the rotating frame. Direction of coriolis force depends on direction of velocity $\vec{v}'$.

Coriolis force $= -2m\vec{\omega} \times \vec{v}' = \vec{F}_{cor}$

or $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$\vec{F}_{cor}$ is $\perp$ to both $\vec{\omega}$ and $\vec{v}'$ and its direction can be found by using right hand thumb rule of cross product of two vectors.
Direction of coriolis force on an insect which is moving along the radius of a rotating disc and along rim is shown in figure.





**Earth: A non-inertial (rotating) frame:** Earth spins about its axis, therefore it is a non-inertial frame.

Angular velocity of earth's spinning motion is $\omega = \frac{2\pi}{T} = \frac{2\pi}{24h} = 7.27 \times 10^{-5}\,\text{rad/sec}$

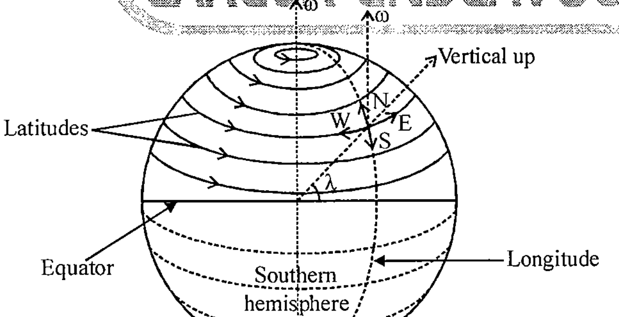
Due to rotations of earth following effects arise.

(a) All objects on earth's surface experience centrifugal force which is directed away from the axis of rotation.

(b) All objects moving on earth's surface experience coriolis force in addition to centrifugal force.

(c) An observer standing on earth's surface notices that on all objects a centrifugal and a coriolis force (if $v' \neq 0$) act irrespective of the location of the object. That is the object which is being observed by the observer may or may not be on the earth's surface. For example we on the earth, see that the sun rises and sets. If we analyse the dynamics of the sun we will find that sun experiences a centrifugal force away from the earth and a coriolis force towards the earth.

**Coordinate system on earth's surface:** To analyse dynamics of objects from earth's surface we define local cartesian coordinate system as follows:
+X-axis : Along local east, –X axis: Along local west
+Y-axis : Along local north, –Y axis: Along local south
+Z-axis : Along local vertical upward, –Z axis: Along local vertical downward
Therefore angular velocity vector of earth's spinning motion lies in Y-Z plane as shown in figure.



A point in northern hemisphere

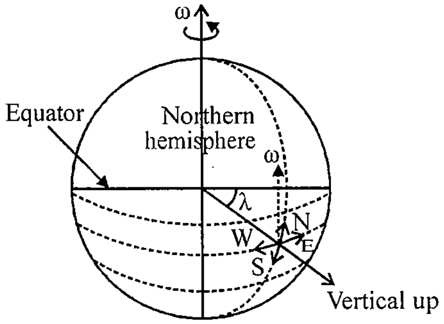
Therefore, at any point in northern hemisphere,

$\vec{\omega} = \omega(\cos\lambda\hat{j} + \sin\lambda\hat{k})$

where $\lambda$ is latitude of the point.

And at any point in southern hemisphere

$\vec{\omega} = \omega(\cos\lambda\hat{j} - \sin\lambda\hat{k})$



A point in southern hemisphere

**Coriolis force on a freely falling object:** An object falls vertically downward therefore velocity of object is in negative z-direction,

$\therefore \vec{v}' = -v'\hat{k}$

In northern hemisphere: $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \;= -2mv'\omega\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k}) \;= +2mv'\omega\cos\lambda\hat{i}$

Coriolic force is directed along +X direction, i.e. towards east. Therefore a freely falling object deviates towards east in northern hemisphere.

In southern hemisphere: $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \;= -2mv'\omega\hat{k} \times (\cos\lambda\hat{j} - \sin\lambda\hat{k}) \;= 2mv'\omega\cos\lambda\hat{i}$

In southern hemisphere also, a freely falling object deviates towards east.

**Coriolis force on river water:** Consider a river flowing from north to south direction in northern hemisphere.

Therefore, velocity of water, $\vec{v}' = -v'\hat{j}$

Coriolis force on water, $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \;= -2mv'\omega\hat{j} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k}) \;= -2mv'\omega\sin\lambda\hat{i}$

Coriolis force on water is directed towards west. Due to which west bank of river flowing from north to south in northern hemisphere undergoes more erosion than east bank.

**Coriolis force on a train moving from west to east:**

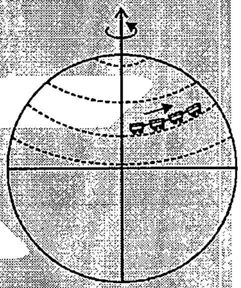
Velocity of train, $\vec{v}' = v'\hat{i}$

Fcoriolis force on train,

$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$= 2mv'\omega\hat{i} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$= 2mv'\omega(\cos\lambda\hat{k} - \sin\lambda\hat{j})$



Therefore lateral force on the track of train due to coriolis force is $2mv\omega\sin\lambda$.

**Coriolis and centrifugal force on the sun as seen from the earth:** The sun appears to be rising and setting because of rotation of earth about its own axis. A person on the earth finds the sun to be moving. This is exactly the same effect as seen by a person sitting in a moving car who sees everything moving past the car.

Let us assume that axis of rotation of earth shown here is perpendicular to the plane of paper. Therefore as seen from the earth the sun appears to be rotating as shown in figure. And velocity of the sun with respect to the earth is

$v' = \omega r$, $r$ = distance of sun from centre of earth

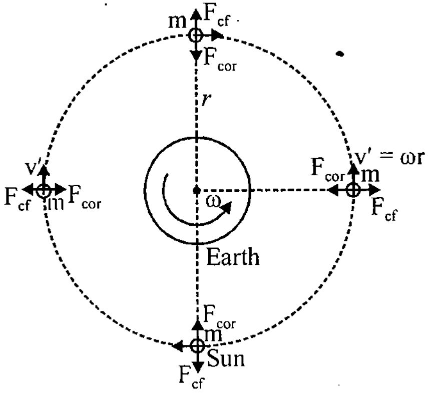
$\therefore$ Centrifugal force is always away from the axis of rotation. The direction of coriolis force can be found using right hand thumb rule and it is found to be towards the centre of earth.

$F_{cor} = |2m\vec{v}' \times \vec{\omega}| = 2mv'\omega\sin 90° = 2mv'\omega$

$= 2m\omega^2 r$

$F_{cf} = m\omega^2 r_\perp,\ r_\perp = r$

$\therefore F_{cf} = m\omega^2 r$



**Eastward deviation of a freely falling object in northern hemisphere:**

Suppose a particle falls from a height $h$ at a place where latitude is $\lambda$.

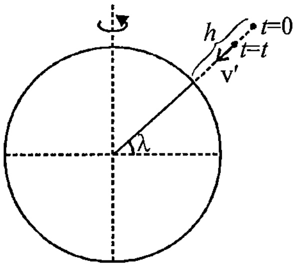
Here we assume that $h \ll R_e$

Velocity of particle at time $t$ is $\vec{v}' = -gt\hat{k}$

Coriolis force on the particle

$$\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega} \quad = -2m\omega gt\hat{k} \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$$

$$= 2m\omega g\cos\lambda t\hat{i}$$



Force is towards east so particle will deviate towards east, equation of motion of the particle in eastward direction is

$$F_{cor} = \frac{md^2x}{dt^2} \quad \text{or} \quad \frac{d^2x}{dt^2} = 2\omega g\cos\lambda t$$

We have following initial condition which we use to solve above equation,

at $t = 0, x = 0, y = 0, z = h$ ... (a)

at $t = 0, \frac{dx}{dt} = 0, \frac{dy}{dt} = 0, \frac{dz}{dt} = 0$ ... (b)

integrating the above differential equation we get, $\frac{dx}{dt} = \omega g\cos\lambda t^2 + c_1$

using (b) we get $c_1 = 0$. Therefore $\frac{dx}{dt} = \omega g\cos\lambda t^2$

Integrating again we get, $x = \frac{1}{3}\omega g\cos\lambda t^3 + c_2$, using (a) we get, $c_2 = 0$

$$\therefore \boxed{x = \frac{1}{3}\omega g\cos\lambda t^3}$$

where $t$ is total time taken to reach earth surface

$$\therefore \; t = \sqrt{\frac{2h}{g}} \qquad \therefore \boxed{x = \frac{1}{3}\omega g\cos\lambda\left(\frac{2h}{g}\right)^{3/2}}$$

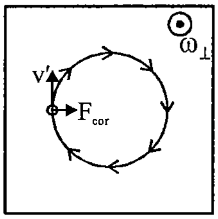
If a particle falls from a height of 100 metre at equator $(\lambda = 0)$ then using above equation we get.

$$x = \frac{1}{3} \times 7.27 \times 10^{-5} \times 9.8 \times \left(\frac{2 \times 100}{9.8}\right)^{3/2} \simeq 2.19 \approx 2.2\text{cm}$$

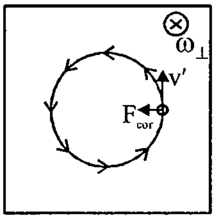
**Note:** The expression derived for eastward deviation is an approximate one. In the derivation we have assumed that particle is uniformly accelerated in downward direction which is not true, when particle attains a velocity in X(each direction) there arises a component of coriolis force in vertical direction, due to which downward acceleration becomes a function of $v_x'$. Because of $v_x'$ component of velocity, particle also experiences a force in southward direction. Therefore a freely falling object in northern hemisphere actually deviates in south-east direction.

**Circular motion on earth's surface due to coriolis force:** Direction of coriolis force is perpendicular to the velocity of particle, therefore in absence of any other force like gravity, friction etc. a particle will move in circular path in the same manner as a charge particle does due to magnetic force.

Let us suppose that there is no friction and a particle is thrown horizontally on the surface of earth with speed $v'$. Let $\lambda$ be the latitude of place. Therefore, $\vec{\omega} = \omega(\cos\lambda\hat{j} \pm \sin\lambda\hat{k})$ we use + or – sign depending on the case whether we are in northern or southern hemisphere. Component of $\vec{\omega}$ perpendicular the plane is $\pm\omega\sin\lambda$. Therefore a particle thrown horizontally will deviate in opposite direction in the two hemispheres, as shown below:



Horizontal plane in N-hemisphere



Horizontal plane in S-hemisphere

Thus, circular path is clock wise in N-hemisphere and anticlockwise in S-hemisphere.

If $r$ be the radius of circle then,

horizontal component of $F_{cor} = \dfrac{mv'^2}{r}$ or $2mv'\omega_\perp \sin 90° = \dfrac{mv'^2}{r}$

or $2mv'\omega\sin\lambda = \dfrac{mv'^2}{r}$ $\quad\therefore \boxed{r = \dfrac{v'}{2\omega\sin\lambda}}$

If the velocity with which particle is thrown is very small say 1mm/sec then radius of circular path near pole $(\lambda = 90°)$ will be

$$r = \frac{1\times10^{-3}}{2\times7.27\times10^{-5}\times1} \simeq 6.87 \text{ metres}$$

A stone thrown horizontally does not move in circle because it falls down due to gravity and then stops because of friction.

## SOLVED PROBLEMS

1. An insect of mass $m$ crawls on earth's surface with velocity $v_0$ at $\phi$ angle north of east. The latitude of place where insect is situated is $\lambda$. If $\omega$ be angular velocity of earth's spinning motion and R be radius of earth. Calculate centrifugal and coriolis force on the insect.

**Soln.** Velocity of insect w.r.t. earth

$$\vec{v}' = v_0(\cos\phi\hat{i} + \sin\phi\hat{j})$$

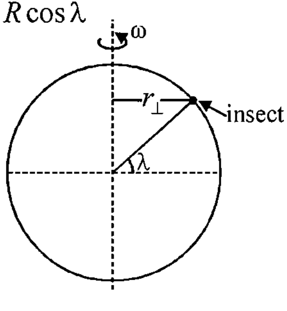
perpendicular distance of insect from axis of rotation is $r_\perp = R\cos\lambda$

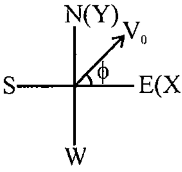
$\therefore$ centrifugal force $= m\omega^2 r_\perp = m\omega^2 R\cos\lambda$

coriolis force $\vec{F}_{cor} = 2m\vec{v}' \times \vec{\omega}$

$= 2mv_0\omega(\cos\phi\hat{i} + \sin\phi\hat{j}) \times (\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$= 2mv_0\omega(\cos\phi\cos\lambda\hat{k} - \cos\phi\sin\lambda\hat{j} + \sin\phi\sin\lambda\hat{i})$





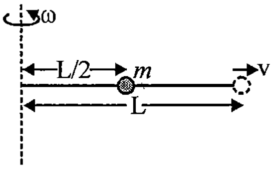
$$\therefore\ \left|\vec{F}_{cor}\right| = 2mv_0\omega\sqrt{\cos^2\phi\cos^2\lambda+\cos^2\phi\sin^2.\lambda+\sin^2\phi\sin^2\lambda}$$

$$= 2mv_0\omega\sqrt{\cos^2\phi+\sin^2\phi.\sin^2\lambda}$$

2. A bead of mass '$m$' lies at the centre of a thin wire of length 'L'. The wire is then rotated with uniform angular velocity $\omega$ about its one end due to which bead slides away from the axis of rotation. Find the velocity of bead with respect to the rod at the moment when the bead is about to leave the free end of the wire.

**Soln.** Due to rotation of the wire the bead is acted upon by centrifugal as well as coriolis force (while moving). However it is centrifugal force which accelerates the bead away from the axis of rotation. Let $v$ be the speed of bead w.r.t. the wire when it leaves the other end. Let us apply work energy theorem to find $v$.

Work done by centrifugal force = change in kinetic energy



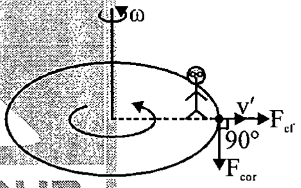
$$\int_{L/2}^{L} m\omega^2 r dr = \frac{1}{2}mv^2 - 0,\quad \frac{m\omega^2}{2}\left[L^2-\frac{L^2}{4}\right]=\frac{1}{2}mv^2$$

$$\therefore v = \frac{\sqrt{3}}{2}\omega L$$

3. A person is standing at the edge of a disc of radius R. The disc is rotating about its axis with uniform angular velocity $\omega$. The person throws a stone in radially outward direction with speed $\frac{\omega R}{2}$ relative for the disc. Calculate acceleration of stone as seen by the person soon after throwing. (neglect gravity).

**Soln.** As seen by the person the stone is acted upon by centrifugal and coriolis forces. The direction of the two forces soon after throwing is shown in the figure.

Net force on the stone is



$$F' = \sqrt{F_{cor}^2 + F_{cf}^2}$$

$$= \sqrt{|2m\vec{v}'\times\vec{\omega}|^2 + (m\omega^2 r_\perp)^2} = \sqrt{\left(2m.\frac{\omega R}{2}.\omega\sin 90^\circ\right)^2 + (m\omega^2 R)^2}$$

$$F' = \sqrt{2}m\omega^2 R$$

Therefore acceleration of stone as seen by the person is

$$a' = \frac{F'}{m} = \sqrt{2}\omega^2 R$$

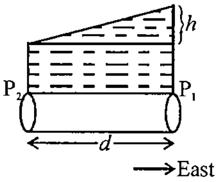
4. Show that when a tidal current runs due north in the northern hemisphere at an altitude of $\lambda$ with velocity $v$ in a channel of width $d$, level of water on east coast is raised by $(2v\omega d\sin\lambda)/g$ above that on the west coast. Here $\omega$ is the angular velocity of the earth.

**Soln.** Water flowing towards the north experiences a coriolis force directed towards east. Due to which water accumulates at the east coast. Thus water level at the east coast is raised above in comparison to the west coast.

In the figure below we show a cross-section of water let its mass be $m$.

$\therefore$ Net coriolis force on it = $2m\omega v\sin\lambda$ (towards east)

This force on the given portion of water is balanced by pressure difference arising due difference in water level



$\therefore\ (P_1 - P_2)A = 2m\omega v \sin\lambda$

$\rho g h A = 2(\rho A d)\omega v \sin\lambda$

$\therefore\ h = 2v\omega d \sin\lambda / g$

5. Show that lateral force on the train rails differ in magnitude by $4mv_0\omega\sin\lambda$ when train travels towards the east and when it travels towards the west with speed $v_0$ and at a latitude $\lambda$.

**Soln.** When train reverses its direction of motion the direction of coriolis force also gets reversed. But the magnitude remains the same. Therefore, magnitude of change in coriolis force is equal to twice the magnitude of coriolis force itself.

$\vec{\omega} = \omega(\cos\lambda\hat{j} + \sin\lambda\hat{k})$

$\vec{v} = v\hat{i}$ (eastward)

$\therefore$ Coriolis force $\vec{F}_{1C} = -2m\vec{\omega}\times\vec{v} = -2m(-\hat{k}\cos\lambda + \hat{j}\sin\lambda)\omega v$

$\vec{v} = -v\hat{i}$ (westward)

$\therefore\ \vec{F}_{2C} = +2m(-\hat{k}\cos\lambda + \hat{j}\sin\lambda)\omega v$

Difference in force = $\vec{F}_{2C} - \vec{F}_{1C} = 2m\omega v\sin\lambda\hat{j}$

lateral force is given by $\hat{j}$ component of coriolis force.

$\therefore$ Difference in lateral force is = $4m\omega v\sin\lambda$

Its direction is northwards.

6. A car running at uniform speed $v_0$ along the equator has a simple pendulum suspended from its roof. Calculate the ratio of time period of small oscillation when the car moves from the west to the east and when it moves from east to west.

**Soln.** Besides gravity, centrifugal and coriolis forces will also act on the pendulum.

$\therefore$ Net force on pendulum when car is going west to east.

$F = mg - m\omega^2 R - 2mv_0\omega$

$\therefore$ effective acceleration of pendulum

$g' = g - \omega^2 R - 2v_0\omega$

$\therefore$ Time period $T_1 = 2\pi\sqrt{\dfrac{l}{g - \omega^2 R - 2v_0\omega}}$

and when car is moving east to west

$g' = g - \omega^2 R + 2v_0\omega$

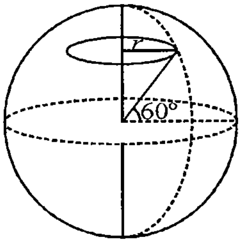
$\therefore$ Time period $T_2 = 2\pi\sqrt{\dfrac{l}{g - \omega^2 R + 2v_0\omega}}$

$\therefore$ ratio of time period $T_1 / T_2 = \sqrt{\dfrac{g - \omega^2 R + 2v_0\omega}{g - \omega^2 R - 2v_0\omega}}$

7. Assuming earth to be a sphere, calculate the linear speed of an object lying on the earth's surface at an altitude of $\lambda = 60°$ and also calculate the centrifugal force experienced by an object of mass 50 kg. Compare this force with gravitational force.

**Soln.** Angular velocity of earth about its axis due to its spinning motion is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{(24\times 60\times 60)} \text{ rad/sec}$$



Radius of the earth R = 6400 × $10^3$ m.

At latitude $\lambda = 60°$ objects move in circle of radius $r = R\cos\lambda$

$\therefore$ linear speed $= \omega r$

$$= \frac{2\pi}{24\times 60\times 60}\times 6400\times 10^3 \cos 60° \text{ m/s}$$

$$= \frac{2\pi\times 10^3}{27} \text{ m/s} \quad \simeq 233 \text{ m/s}$$

Centrifugal force = $m\omega^2 r$

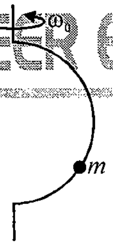
$$= 50\times\left(\frac{2\pi}{24\times 60\times 60}\right)^2 \times 6400\times 10^3$$

$$\simeq 1.70\text{N}$$

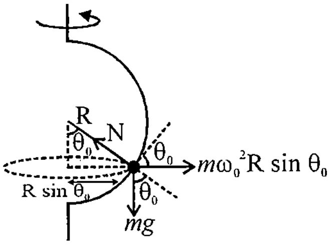
Gravitational force = $mg$ = 50 × 9.8 = 490N

Since centrifugal force due to spinning of earth is very small compared to the gravitational force due to this reason we do not feel the rotation of earth.

8. A semi-circular wire rotating with uniform angular velocity $\omega_0$ has a small bead of mass $m$ threaded into it. Calculate the time period of small oscillation of bead about the equilibrium point.



**Soln.** At equilibrium position, tangential component of weight is balanced by tangential component of centrifugal force.



$$\therefore \; mg\sin\theta_0 = m\omega_0^2 R\sin\theta_0 \cdot \cos\theta_0$$

$$\therefore \; \cos\theta_0 = \frac{g}{\omega_0^2 R}$$

$$\text{or } \theta_0 = \cos^{-1}\left[\frac{g}{\omega_0^2 R}\right] \qquad \ldots (i)$$

Now, let the particle is displaced by angle $\theta$ (which is very small) from its equilibrium position.

Therefore, now line joining the particle to the centre of wire makes $(\theta_0+\theta)$ with the vertical.

Therefore, net torque on the bead (particle)

$$= mg\cdot R\sin(\theta+\theta_0) - m\omega_0^2 R\sin(\theta+\theta_0)\cdot R\cos(\theta+\theta_0)$$

$$= m\omega_0^2R^2\sin(\theta_0+\theta)\left\{\frac{g}{\omega_0^2R}-\cos(\theta+\theta_0)\right\}$$

$$= m\omega_0^2R^2\sin(\theta_0')\left\{\frac{g}{\omega_0^2R}-\cos\theta\cdot\cos\theta_0+\sin\theta\cdot\sin\theta_0)\right\}$$

from (i) $$= m\omega_0^2R^2\sin\theta_0\left\{\frac{g}{\omega_0^2R}-1\cdot\frac{g}{\omega_0^2R}+\theta\cdot\sin\theta_0\right\} = m\omega_0^2R^2\sin^2\theta_0\cdot\theta$$

This is the restoring torque

$\therefore \quad -mR^2\ddot{\theta} = mR^2\omega_0^2\sin^2\theta_0\cdot\theta$

or $\quad \ddot{\theta} = -\omega_0^2\sin^2\theta_0\cdot\theta$

or $\quad \ddot{\theta}+\omega_0^2\sin^2\theta_0\cdot\theta = 0$

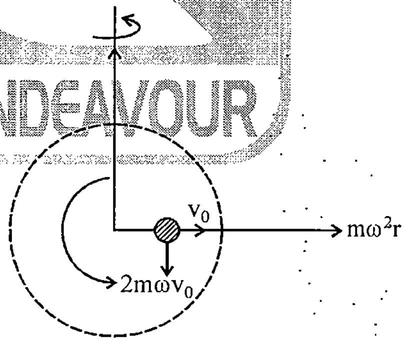
This is the equation of S.H.M. with frequency

$$\omega = \omega_0\sin\theta_0 = \omega_0\sqrt{1-\cos^2\theta_0} \quad\Rightarrow \omega = \omega_0\sqrt{1-\left(\frac{g}{\omega_0^2R}\right)^2}$$

9. A particle of mass m moves radially outward with a constant velocity $v_0$, on a turn table rotating with uniform angular velocity ω. Find co-efficients of friction.

**Soln.** On the turn table particle experiences coriolis and centrifugal forces.

$$\vec{F}_{\text{coriolis}} = -2m\,\vec{\omega}\times\vec{v}_0$$
$$= -2m\,\omega\,v_0\,\hat{z}\times\hat{r}$$
$$= -2m\,\omega\,v_0\,\hat{\theta}$$



$$\vec{F}_{\text{centrifugal}} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$= -m\,\vec{\omega}\times(\omega\,r\,\hat{\theta})$$
$$= m\,\omega^2\,\hat{r}$$

Net force on the particle

$$F = \sqrt{4m^2\omega^2v_0^2+m^2\omega^4r^2} \quad\Rightarrow F = m\,\omega\,v_0\sqrt{4+\left(\frac{\omega\,r}{v_0}\right)^2}$$

Since, $v_0$ is constant, the net force is balanced by friction force.

$$\therefore\quad f_r = F$$

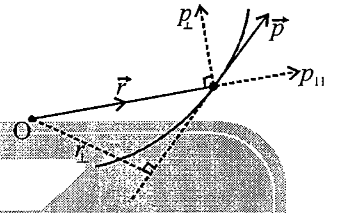
$$\mu mg = m\,\omega\,v_0\sqrt{4+\left(\frac{\omega\,r}{v_0}\right)^2}$$

$$\therefore\quad \mu = \frac{\omega\,v_0}{g}\sqrt{4+\left(\frac{\omega\,r}{v_0}\right)^2}$$

# Chapter 7

# *Angular momentum and torque*

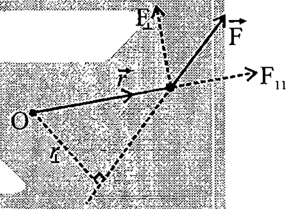
**Angular momentum:** If $\vec{p}$ be momentum of a particle and $\vec{r}$ be its position vector w.r.t. a point 'o' then angular momentum about point 'o' is defined as



$$\vec{L} = \vec{r}\times\vec{p}$$

Magnitude of angular momentum is written as $L = rp_\perp$ or $L = r_\perp p$

**Torque:** If $\vec{F}$ be force acting at some point and $\vec{r}$ be the position vector of the point w.r.t. some origin 'O' then torque about 'O' is defined as $\vec{\tau} = \vec{r}\times\vec{F}$



Magnitude of force is written as $\tau = rF_\perp$ or $\tau = r_\perp F$

If a system is in translational equilibrium then net force is zero however if system is in rotational equilibrium then net torque is zero about any point.

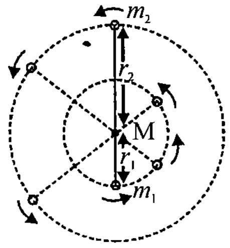
**Relation between torque and angular momentum:**

$$\vec{L} = \vec{r}\times\vec{p}$$

Differentiate w.r.t. time to get $\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$

If $\vec{\tau} = 0$ then $\vec{L}$ = constant. This is called conservation of angular momentum.

**Double star problem:** A double star consists of two stars revolving in circular orbits about their common centre of mass. The necessary centripetal force for circular motion is provided by their mutual gravitational attraction. Let us consider a double star which consists of two star of masses $m_1$ and $m_2$ separated by a distance $r$.



There is no external force on the system therefore centre of mass of the system does not move.

Let $r_1$ and $r_2$ be radii of the orbits of the two stars.

$$\therefore\quad r_1 = \frac{m_2 r}{m_1 + m_2} \Rightarrow \frac{r_1}{r_2} = \frac{m_2}{m_1},$$

$$r_2 = \frac{m_1 r}{m_1 + m_2}$$

Since centre of mass is stationary. Therefore, the angular velocity of the two stars must be same.

$\therefore$ For star 1 $\quad m_1\omega^2 r_1 = \dfrac{Gm_1.m_2}{r^2} \Rightarrow \omega = \sqrt{\dfrac{G(m_1+m_2)}{r^3}}$

For start 2 $\quad m_2\omega^2 r_2 = \dfrac{Gm_1.m_2}{r^2} \Rightarrow \omega = \sqrt{\dfrac{G(m_1+m_2)}{r^3}}$

Ratio of linear speeds of two stars:

$$\frac{v_1}{v_2} = \frac{\omega r_1}{\omega r_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1} \qquad \therefore \boxed{\frac{v_1}{v_2} = \frac{m_2}{m_1}}$$

Ratio of kinetic energy of two stars

$$\frac{K_1}{K_2} = \frac{\frac{1}{2}m_1v_1^2}{\frac{1}{2}m_2v_2^2} = \frac{m_1(\omega r_1)^2}{m_2(\omega r_2)^2} = \frac{m_1}{m_2}.\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2}{m_1} \qquad \therefore \boxed{\frac{K_1}{K_2} = \frac{m_2}{m_1}}$$

Ratio of angular momentum of two stars

$$\frac{L_1}{L_2} = \frac{m_1v_1r_1}{m_2v_2r_2} = \frac{m_1\omega r_1^2}{m_2\omega r_2^2} = \frac{m_2}{m_1} \qquad \therefore \boxed{\frac{L_1}{L_2} = \frac{m_2}{m_1}}$$

ratio of magnitude of linear momentum $\dfrac{P_1}{P_2} = \dfrac{m_1v_1}{m_2v_2} = \dfrac{m_1\omega r_1}{m_2\omega r_2} = \dfrac{m_1}{m_2}.\dfrac{r_1}{r_2} = \dfrac{m_1}{m_2}.\dfrac{m_2}{m_1} \quad \therefore \boxed{\dfrac{P_1}{P_2} = 1}$

Total energy of double star system $E = \dfrac{1}{2}m_1v_1^2 + \dfrac{1}{2}m_2v_2^2 - \dfrac{Gm_1.m_2}{r} = \dfrac{1}{2}m_1\omega^2r_1^2 + \dfrac{1}{2}m_2\omega^2r_2^2 - \dfrac{Gm_1.m_2}{r}$

put values of $r_1, r_2$ and $\omega$ to get

$$E = \frac{1}{2}\frac{G(m_1+m_2)}{2r^3}\left[\frac{m_1m_2^2r^2 + m_1^2r^2m_2}{(m_1+m_2)^2}\right] - \frac{Gm_1m_2}{r} = \frac{Gm_1.m_2}{2r} - \frac{Gm_1.m_2}{r} \qquad \boxed{E = \frac{-Gm_1.m_2}{2r}}$$

## SOLVED PROBLEMS

1. Three particles are moving in $x$-$y$ plane with constant velocities as shown in the figure. Calculate angular momentum of each particle about the origin.

**Soln.** Angular momentum can be calculated in various ways.

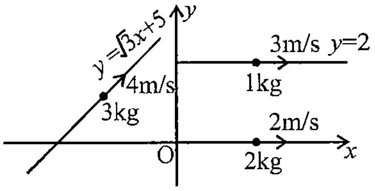
Let us use $L = r_\perp p$

For 1 kg particle:

momentum $(p) = mv = 1\times 3 = 3\,\text{kg m/s}$

perpendicular distance of line of momentum from origin $(r_\perp) = 2m$

$\therefore L = r_\perp p = 2\times 3 = 6\,\text{kg m}^2/\text{s}$



For 2kg particle: Line of momentum of this particle passes through origin therefore $r_\perp = 0 \;\therefore L = 0$

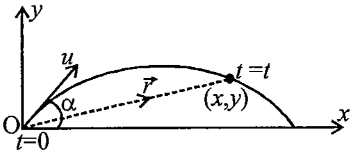
For 3kg particle: $r_\perp = \dfrac{|y - \sqrt{3}x - 5|_{x=0,y=0}}{\sqrt{1^2 + (\sqrt{3})^2}} = \dfrac{|0 - \sqrt{3}\times 0 - 5|}{\sqrt{4}} = \dfrac{5}{2}$

$p = mv = 3\times 4 = 12\text{kg m/s}$

$\therefore \text{L} = r_\perp p = \dfrac{5}{2}\times 12 = 30\text{kg m}^2/\text{sec}$

2. A particle of mass $m$ is thrown with initial speed $u$ at angle $\alpha$ with horizontal. If maximum height reached is much small compared to radius of earth. Calculate angular momentum of particle about point of projection after time $t$.

**Soln.** Position vector of particle after time $t$



$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{r} = u\cos\alpha t\hat{i} + (u\sin\alpha t - \frac{1}{2}gt^2)\hat{j}$$

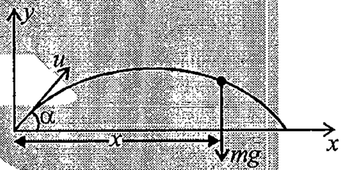
velocity of particle after time $t$

$$\vec{v} = \frac{d\vec{r}}{dt} = u\cos\alpha\hat{i} + (u\sin\alpha - gt)\hat{j}$$

$$\therefore \vec{L} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} \quad = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u\cos\alpha t & u\sin\alpha t - \frac{1}{2}gt^2 & 0 \\ u\cos\alpha & u\sin\alpha - gt & 0 \end{vmatrix}$$

$$= \hat{k}[u^2\cos\alpha\sin\alpha t - gt^2u\cos\alpha - u^2\cos\alpha\sin\alpha t + \frac{1}{2}gu\cos\alpha t^2] = -\frac{1}{2}gt^2u\cos\alpha\hat{k}$$

**Another method:** At time $t$, torque on the particle about point of projection is



$$\tau = r_\perp F = xmg$$

or $\vec{\tau} = -mg\,x\,\hat{k}$

$$\vec{\tau} = -mg\,u\cos\alpha t\,\hat{k}$$

We know that $\vec{\tau} = \frac{d\vec{L}}{dt}$

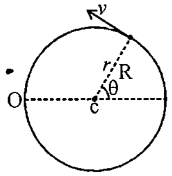
$$\therefore \quad \frac{d\vec{L}}{dt} = -mgu\cos\alpha t\hat{k} \quad \text{or} \quad \int_0^{\vec{L}} d\vec{L} = -mgu\cos\alpha\int_0^t tdt\hat{k}$$

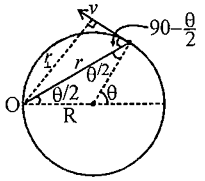
$$\therefore \quad \vec{L} = -\frac{1}{2}mgut^2\cos\alpha\,\hat{k}$$

3. A particle of mass '$m$' moves with constant speed $v$ along a circle of radius R. Calculate angular momentum about centre of circle. Also calculate angular momentum about a point on the circumference of circle.

**Soln.** About centre $L_c = r_\perp . p = Rmv$

About a point on circumference



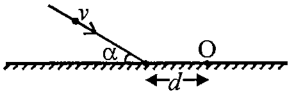


$$L_O = r_\perp p \; = r\sin\left(90 - \frac{\theta}{2}\right)mv \text{ (from figure)}$$

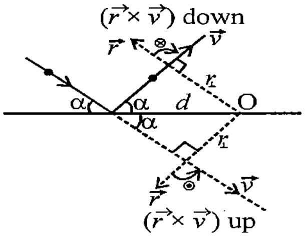
$$= r\cos\frac{\theta}{2}mv \; = \left(2R\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2}mv \text{ (use sine rule to get } r = 2R\cos\frac{\theta}{2}\text{)}$$

$$= 2Rmv\cos^2\frac{\theta}{2}$$

4. A particle of mass *m* moving with constant speed *v* strikes a floor elastically. Calculate change in angular momentum about a point which is at a distance '*d*' from the point of impact as shown in figure.



**Soln.** The collision is elastic therefore particle will rebound with same speed and at same angle with the floor as shown in the figure.



$\therefore r_\perp = d\sin\alpha$

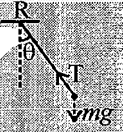
Angular momentum before impact

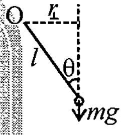
$L_1 = r_\perp p = d\sin\alpha\, mv,\quad L_2 = r_\perp p = -d\sin\alpha\, mv$

$\therefore$ Change in angular momentum $\Delta L = L_2 - L_1 = -2mvd\sin\alpha$

5. A simple pendulum consisting of a small bob of mass *m* and light string of length '*l*' oscillates in vertical plane. Calculate torque due to gravity and tension in the string at the instant string makes $\theta$ angle with downward vertical.

**Soln.** Line of action of tension passes through point of suspension. Therefore $r_\perp = 0$





Thus, torque due to tension = 0

For *mg*, $r_\perp = l\sin\theta$

$\therefore \tau = r_\perp F$

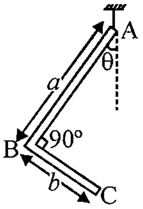
$= l\sin\theta mg = mgl\sin\theta$

6. An L-shaped uniform wire is suspended from a string as shown in figure. If length of the two sections of wire be '*a*' and '*b*' then what is value of angle $\theta$ made by upper section with vertical.

**Soln.** The wire is in rotational equilibrium therefore centre of mass of the rod must lie on the vertical line through point A so that gravity does not produce any torque about point A.

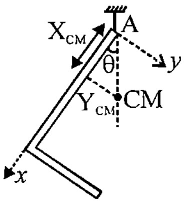
To find the position of centre of mass let us choose coordinate system as shown in figure.

Let $\lambda$ be mass per unit length of the wire.



$$\therefore X_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{\lambda a.\frac{a}{2} + \lambda ba}{\lambda a + \lambda b} = \frac{a(a+2b)}{2(a+b)}$$

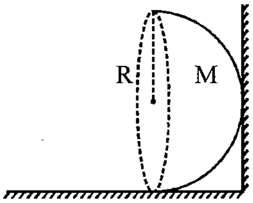
$$Y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{\lambda a.0 + \lambda b.\frac{b}{2}}{\lambda a + \lambda b} = \frac{b^2}{2(a+b)}$$



$$\therefore \tan\theta = \frac{Y_{CM}}{X_{CM}} = \frac{b^2}{a(a+2b)}$$

or $\theta = \tan^{-1}\left(\frac{b^2}{a(a+2b)}\right)$ if $a = b$ then $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

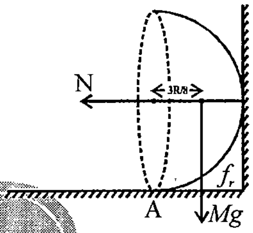
7. A solid hemisphere of mass M and radius R is placed against a smooth wall as shown in the figure. What is normal reaction on the sphere due to the wall.



**Soln:** Centre of mass of the hemisphere lies at a distance $\frac{3R}{8}$ from its base centre. The sphere is in equilibrium therefore torque about point A must be zero.

$$\therefore NR - Mg.\frac{3R}{8} = 0$$

or, $N = \frac{3Mg}{8}$



8. A thin rod of mass M and length L is suspended from one end while its other end lies on a smooth horizontal plane and makes an angle $\alpha$ with the plane. Find the normal reaction from the plane and tension in the string from which it is suspended.

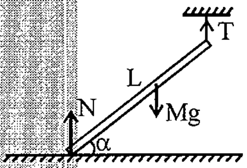
For translational equilibrium $T + N = Mg$ ... (a)

For rotational equilibrium

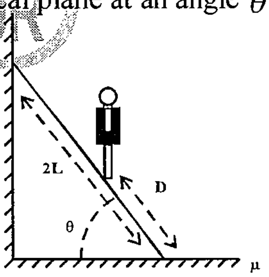
Torque about lower end of the rod is zero

$$\therefore \; T.L\cos\alpha - Mg\frac{L}{2}\cos\alpha = 0 \qquad \therefore \; T = \frac{Mg}{2}$$

from (a) $N = Mg - T = Mg - \frac{Mg}{2} = \frac{Mg}{2}$



9. A uniform ladder of length 2L and mass 'm' leans against a wall in a vertical plane at an angle $\theta$ to the horizontal. The floor is rough, having a coefficient of static friction μ. A person of mass M stands on the ladder at a distance D from its base (see figure). If the wall is frictionless, the maximum distance ($D_{max}$) up the ladder that the person can reach before the ladder slips is



**Soln.** Let us take ladder plus person as system. Various forces acting on the system are shown in the figure. For translational equilibrium we must have

$N_1 = (M + m)g$ ... (a)

$N_2 = f_r$ ... (b)

For rotational equilibrium torque about lower end of the ladder must be zero. Therefore,

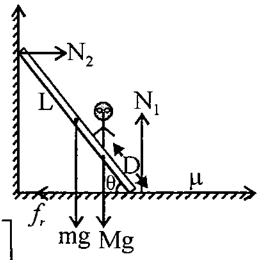
$$MgD\cos\theta + mgL\cos\theta - N_2.2L\sin\theta = 0$$

$$\therefore N_2 = \frac{g\cot\theta}{2}\left[\frac{MD}{L} + m\right]$$

$\therefore$ from (b) we get, $f_r = \dfrac{g\cot\theta}{2}\left[\dfrac{D}{L}M + m\right]$

since $f_r \le \mu N_1$ , $\therefore \dfrac{g\cot\theta}{2}\left[\dfrac{D}{L}M + m\right] \le \mu(M+m)g$

or $D \le \dfrac{L}{M}[2\mu(M+m)\tan\theta - m]$ $\therefore D_{\max} = L\left[2\mu\left(1+\dfrac{m}{M}\right)\tan\theta - \dfrac{m}{M}\right]$



10. Angular momentum of an object about a point varies with time as $\vec{L} = \alpha\hat{i} + \beta t^2\hat{j}$. At a certain instant of time, torque acting on the object about that point makes 60° with $\vec{L}$. Calculate magnitude of torque at that instant.

**Soln.** Torque is related to angular momentum as

$$\vec{\tau} = \frac{d\vec{L}}{dt}, \quad \vec{\tau} = \frac{d}{dt}(\alpha\hat{i} + \beta t^2\hat{j})$$

$$\vec{\tau} = 2\beta t\hat{j} \qquad \text{... (i)}$$

given, $\theta = \cos^{-1}\left[\dfrac{\vec{L}\cdot\vec{\tau}}{|\vec{L}||\vec{\tau}|}\right] = 60^\circ \Rightarrow \cos^{-1}\left[\dfrac{2\beta^2 t^3}{2\beta t(\alpha^2+\beta t^2)^{1/2}}\right] = 60^\circ$

or $\dfrac{2\beta^2 t^3}{2\beta t(\alpha^2+\beta^2 t^4)^{1/2}} = \dfrac{1}{2}$

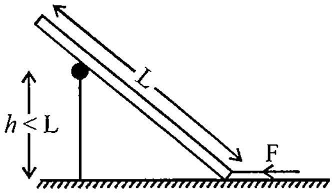
or $\beta^2 t^4 = \dfrac{1}{4}(\alpha^2 + \beta^2 + 4)$ or $4\beta^2 t^4 - \beta^2 t^4 - \alpha^2 = 0$

$3\beta^2 t^4 = \alpha^2$ or $t = \left[\dfrac{\alpha^2}{3\beta^2}\right]^{1/4}$

$\therefore$ from (i)

$$\therefore \tau = 2\beta\left[\frac{\alpha^2}{3\beta^2}\right]^{1/4}\hat{j}$$

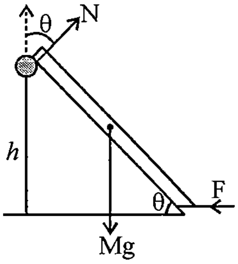
11. Consider the situation shown in the figure. All the surfaces are smooth and the rod has mass M.



What is the minimum value of F required to keep the rod in equilibrium?

**Soln.** Clearly, the force will be minimum when upper end of the rod lies on the round support. We are excluding the case when rod is vertical.

In figure, $\sin\theta = \dfrac{h}{L} \Rightarrow \tan\theta = \dfrac{h}{\sqrt{L^2 - h^2}}$



$N\cos\theta = Mg$

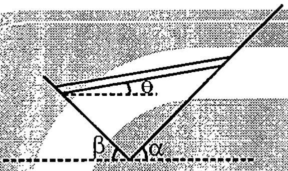
$N\sin\theta = F$

$\therefore\ \tan\theta = \dfrac{F}{Mg}$

or $F = Mg\tan\theta$

$$F = Mg \cdot \frac{h}{\sqrt{h^2 - L^2}}$$

12. A thin rod of mass M and length L is lying on two inclined smooth planes having angles of inclination $\alpha$ and $\beta$ as shown in the figure. Calculate the angle $\theta$ that the rod makes with horizontal in equilibrium.



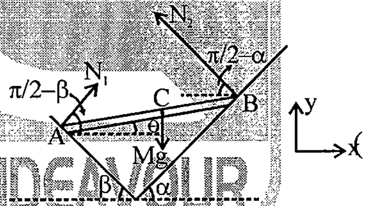
**Soln.** The forces acting on the rod are shown in the figure.

In equilibrium,

$\Sigma F_x = 0, \Sigma F_y = 0, \Sigma\tau = 0$

From $\Sigma F_x = 0$, we get

$N_1\sin\beta = N_2\sin\alpha$



or $\quad N_1 = \dfrac{N_2\sin\alpha}{\sin\beta}$ ... (i)

From $\Sigma F_y = 0$, we get

$N_1\cos\beta + N_2\cos\alpha = Mg$

using (1) we get

$$\frac{N_2\sin\alpha\cdot\cos\beta}{\sin\beta} + N_2\cos\alpha = Mg$$

or $N_2 = \dfrac{Mg\cdot\sin\beta}{\sin(\alpha+\beta)}$ ... (ii)

Now, from $\Sigma\tau_A = 0$, we get

$\overrightarrow{AB}\times\overrightarrow{N}_2 + \overrightarrow{AC}\times(-Mg\hat{j}) = 0$ ... (iii)

Here $\overrightarrow{AB} = L(\cos\theta\hat{i} + \sin\theta\hat{j})$

$\overrightarrow{N}_2 = N_2(-\sin\alpha\hat{i} + \cos\alpha\hat{j})$

$$\overrightarrow{AC}=\frac{L}{2}(\cos\theta\hat{i}+\sin\theta\hat{j})$$

$\therefore$ from (iii) we get

$$LN_2(\cos\alpha\cos\theta+\sin\theta\sin\alpha)\hat{k}-\frac{MgL}{2}\cos\theta\hat{k}=0$$

use $N_2$ to get

$$\text{or}\quad\left[\frac{1}{2}-\frac{\sin\beta\cos\alpha}{\sin(\alpha+\beta)}\right]\cos\theta+\frac{\sin\alpha\cdot\sin\beta}{(\sin\alpha+\beta)}\sin\theta=0$$

$$\text{or}\quad\tan\theta=\frac{2\sin\beta\cdot\cos\alpha-\sin(\alpha+\beta)}{2\sin\alpha\cdot\sin\beta)}\;=\frac{1}{2}(\cot\alpha-\cot\beta)$$

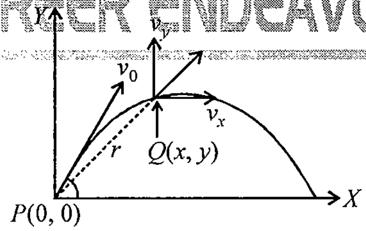
$$\therefore\;\theta\;=\;\tan^{-1}\left[\frac{1}{2}(\cot\alpha-\cot\beta)\right]$$

13. A particle is projected at time $t=0$ from a point $P$ on the ground with a speed $v_0$ at an angle 45° to the horizontal. Find the magnitude and direction of the angular momentum of the particle about the point $P$, at time $t=\frac{v_0}{g}$.

**Soln.** In figure, $P$ is the origin (0, 0) from where the particle is projected with velocity $v_0$ at time $t=0$. Let after time $t$, $Q(x, y)$ be the co-ordinates of the particle, where the resultant velocity of the particle is $v$. The distance of $Q$ from $P$ is $r$.

The rectangular components $v_x$ and $v_y$ along the $x$ and $y$ axes are $v_x=v_0\cos 45°$ and $v_y=v_0\sin 45°-gt$.

The co-ordinates $(x, y)$ of the point $Q$ after time $t$ are given by $x=v_0t\cos 45°$ and $y=v_0t\sin 45°-\frac{1}{2}gt^2$.



Angular momentum of particle $Q(x, y)$ about point $P(0, 0)=mv_yx-mv_xy$

Substituting the values, we get

$$=m(v_0\sin 45°-gt)(v_0t\cos 45°)-m(v_0\cos 45°)(v_0t\sin 45°-\frac{1}{2}gt^2)$$

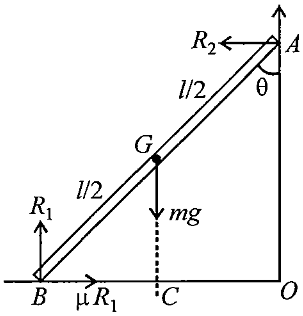
$$=-\frac{1}{2\sqrt{2}}\frac{mv_0^3}{g}.$$

Solving it we get the magnitude of angular momentum of the particle about $P=\frac{1}{2\sqrt{2}}\frac{mv_0^3}{g}$.

The direction of angular momentum will be mutually perpendicular to distance $r$ and momentum vector. The direction will be along the inward normal to the plane of the paper.

14. A uniform ladder $AB$ of length 3 meters and weight 20 kg. rests with the end A against rough vertical wall and the end B on level ground. If the wall and the ground are equally rough and the coefficient of friction is 0.5, find the limiting position of equilibrium.

**Soln.** The different forces on the ladder are shown in figure. Let in the limiting position of equilibrium, the ladder makes an angle $\theta$ with the wall.



Equating the horizontal and vertical forces, we get

$$R_2 = \mu R_1 \quad \text{...(i)}$$

$$R_1 + \mu R_2 = mg \quad \text{...(ii)}$$

From equating (ii),

$$R_1 + \mu^2 R_1 = mg \text{ or } R_1 = \frac{mg}{(1+\mu^2)} \quad \text{...(iii)}$$

From equation (i),

$$R_2 = \left(\frac{\mu}{1+\mu^2}\right) mg \quad \text{...(iv)}$$

Taking the moment of forces about $B$, we get

$$mg \times BC = R_2 \times OA + \mu R_2 \times OB.$$

From figure,

$$OA = l\cos\theta,\ OB = l\sin\theta \text{ and } BC = \frac{l}{2}\sin\theta$$

$$\therefore \quad mg\frac{l}{2}\sin\theta = R_2\, l\cos\theta + \mu R_2\, l\sin\theta$$

$$\sin\theta\left[\frac{mg}{2} - \mu R_2\right] = R_2\cos\theta$$

$$\tan\theta = \frac{R_2}{\left[\frac{mg}{2} - \mu R_2\right]} = \frac{\left(\frac{\mu}{1+\mu^2}\right)mg}{\left[\frac{mg}{2} - \frac{\mu^2}{1-\mu^2}mg\right]}$$

$$\tan\theta = \frac{2\mu}{1-\mu^2} = \frac{2\times 0.5}{1-(0.5)^2} = 1.33 \quad (\because \mu = 0.5)$$

$$= \tan^{-1}(1.33)$$

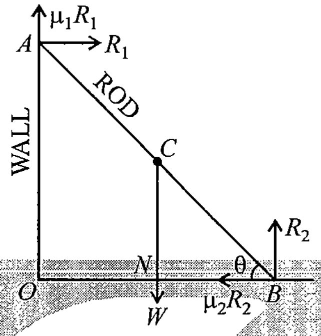
$$\theta = 53^\circ 4'.$$

15. A uniform rod is made to lean between a rough vertical wall and the ground. Show that the least angle at which the rod can be leaned without slipping is given by

$$\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right),$$

where $\mu_1$ and $\mu_2$ stand for the coefficient of friction between (i) the rod and the wall, and (ii) the rod and the ground.

**Soln.** The situation is shown in figure.



Considering the equilibrium of the rod, we have

$$R_1 + \mu_2 R_2 \qquad \text{...(i)}$$

and $$R_2 + \mu_1 R_1 = W \qquad \text{...(ii)}$$

Taking the moments of the forces about $A$, we have

$$R_2 \times OB = W \times ON + \mu_2 R_2 \times OA$$

$$R_2 \times AB\cos\theta = W \times \frac{AB\cos\theta}{2} + \mu_2 R_2 \times AB\sin\theta$$

$$\cos\theta\,(R_2 - \frac{W}{2}) = \sin\theta\,(\mu_2 R_2).$$

$$\therefore \quad \tan\theta = \frac{(R_2 - \frac{W}{2})}{\mu_2 R_2} \qquad \text{...(iii)}$$

From equations (i) and (ii), we have

$$W = R_2 + \mu_1 R_1 = R_2 + \mu_1\mu_2 R_2 = R_2(1+\mu_1\mu_2)$$

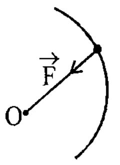
Now $$R_2 - \frac{W}{2} = R_2 - \frac{R_2}{2}[(1+\mu_1\mu_2)] = R_2\left(1-\frac{1}{2}-\frac{\mu_1\mu_2}{2}\right) = \frac{R_2}{2}(1-\mu_1\mu_2).$$

$$\therefore \quad \tan\theta = \frac{\frac{R_2}{2}(1-\mu_1\mu_2)}{\mu_2 R_2} = \frac{1-\mu_1\mu_2}{2\mu_2}$$

$$\theta = \tan^{-1}\left(\frac{1-\mu_1\mu_2}{2\mu_2}\right).$$

# Chapter 8

## Central force motion and Gravitation



A central force is expressed as $\vec{F} = \pm f(r)\hat{r}$, positive sign is used for repulsive and negative sign for attractive force. Central force acting on a particle is always directed towards a point which is called centre of force.

**Features of central force motion:**

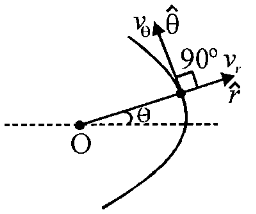
- Central forces are conservative, therefore total energy of a particle moving under central force is conserved.
- Torque about centre of force is zero, therefore angular momentum about centre of force is conserved. This implies that central force motion is a planar motion.
- Areal velocity is constant, that is area swept by line joining the particle to the centre of force per unit time is constant.

$$\frac{\Delta A}{\Delta t} = \text{constant}$$

**Relation between angular momentum and areal velocity:** Central force motion is a planar motion, therefore we can use plane polar coordinate to describe dynamics of particle.

Therefore velocity of particle is $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

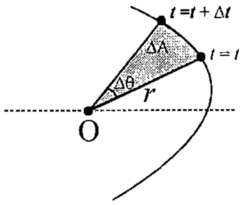
Angular momentum of particle about centre of force 'O' is



$$L = mv_\theta r = mr\dot{\theta}r = mr^2\dot{\theta}$$

$$\boxed{L = mr^2\dot{\theta}}$$

**Areal velocity:** $\dfrac{\Delta A}{\Delta t} = \dfrac{\frac{1}{2}r^2\Delta\theta}{\Delta t} = \dfrac{1}{2}r^2\dfrac{\Delta\theta}{\Delta t} = \dfrac{1}{2}r^2\dot{\theta}$



from expression of angular momentum $r^2\dot{\theta} = \dfrac{1}{m}$

$$\therefore \boxed{\text{Areal velocity} = \frac{L}{2m}}$$

**Differential equation of orbit:** Acceleration of a particle moving in a plane can be expressed in plane polar coordinate as

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Therefore in central force motion Newton's equation is

$\vec{F} = m\vec{a}$, therefore, $f(r)\hat{r} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}]$

Therefore, equating coefficient of $\hat{r}$ and $\hat{\theta}$ we get

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \qquad \text{... (i)}$$

and $m(\ddot{r} - r\dot{\theta}^2) = f(r)$ ... (ii)

Equation (i) can also be obtained by differentiating $L = mr^2\dot{\theta}$ therefore it implies constancy of angular momentum. Equation (ii) is differential equation for the dynamics of particle under central force. To obtain differential equation of orbit we must remove time from equation (ii). To accomplish this we introduce a new variable $u = 1/r$ and use $\dot{\theta} = L/mr^2 = Lu^2/m$. This leads to,

$$\boxed{\frac{\partial^2 u}{\partial\theta^2} + u = -\frac{mf(r)}{L^2u^2}} \qquad \text{... (iii)}$$

equation (iii) is differential equation of orbit. The above equation is mostly used for following purposes.

(a) To find $f(r)$ when equation of orbit is given. That is a relation between $r$ and $\theta$ is given then $f(r)$ can be found.

(b) If $f(r)$ is given then to find equation of orbit.

**Example:** Equation of the orbit of a particle moving under central force is $r\theta = \beta$, where $\beta$ is a constant. Find the force acting on the particle.

**Soln.** $r\theta = \beta$, therefore, $u = \frac{1}{r} = \frac{\theta}{\beta}$ $\therefore \frac{\partial^2 u}{\partial\theta^2} = 0$

Differential equation of orbit, $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-mf(r)}{L^2u^2}$

$\therefore\ 0 + u = \frac{-mf(r)}{L^2u^2}$ $\qquad \therefore f(r) = \frac{-L^2u^3}{m} = \frac{-L^2}{mr^3}$

**Example:** Equation of orbit of a particle moving under central force is $r^n = a\cos n\theta$. Find the force on the particle.

**Soln.** Given, $r^n = a\cos n\theta$, therefore, $u^n = \frac{1}{a\cos n\theta} = \frac{1}{a}\sec n\theta$ ... (a)

Taking ln both sides we get, $n\ln u = \ln\left(\frac{1}{a}\right) + \ln\sec n\theta$

Differentiating w.r.t. $\theta$ we get, $\frac{n}{u}\frac{\partial u}{\partial\theta} = n\tan n\theta$ $\qquad \therefore \frac{\partial u}{\partial\theta} = u\tan n\theta$

Differentiating again w.r.t. $\theta$ we get $\frac{\partial^2 u}{\partial\theta^2} = \frac{\partial u}{\partial\theta}\tan n\theta + u\,n\sec^2 n\theta$ $= u\tan^2 n\theta + u\,n\sec^2 n\theta$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-mf(r)}{L^2u^2}$

$$u\tan^2 n\theta + u\,n\sec^2 n\theta + u = \frac{-mf(r)}{L^2u^2},\quad u\sec^2 n\theta + u\,n\sec^2 n\theta = \frac{-mf(r)}{L^2u^2}$$

$$\therefore f(r) = -\frac{L^2u^3(1+n)\sec^2 n\theta}{m}$$

from (a) $\sec^2 n\theta = a^2u^{2n}$

$$\therefore \qquad f(r) = \frac{-L^2(n+1)a^2u^{2n+3}}{m}$$

$$\therefore \qquad f(r) = \frac{-L^2(n+1)a^2}{m}.\frac{1}{r^{2n+3}}$$

Or $\boxed{f(r) \propto \frac{1}{r^{2n+3}}}$, the force is attractive in nature.

**Example:** A particle of mass $m$ is moving under a central force. The angular momentum of the particle is L and its equation of orbit is $r = Ae^{k\theta}$. Calculate potential energy of the particle.

**Soln.** Potential energy is given as $V(r) = -\int f(r)dr$ ... (a)

Therefore, let us first find $f(r)$.

Given $r = Ae^{k\theta}$, therefore, $u = \frac{1}{A}e^{-k\theta}$ $\therefore \frac{\partial^2 u}{\partial\theta^2} = k^2u$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-mf(r)}{L^2u^2}$ $\therefore (k^2+1)u = \frac{-mf(r)}{L^2u^2}$

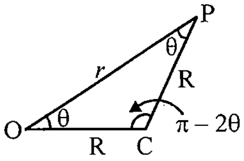
$\therefore f(r) = \frac{-L^2(k^2+1)u^3}{m}$ or $f(r) = \frac{-L^2(k^2+1)}{mr^3}$

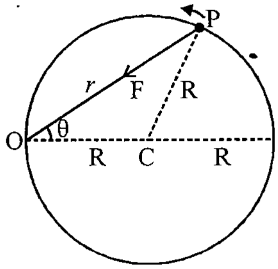
Therefore, from (a) $V(r) = -\int f(r)dr = \frac{-L^2(k^2+1)}{2mr^2}$

**Example:** A particle is moving in a circle under a force which is always directed towards a point on the circumference of the circle. Find the force law.

**Soln.** Let R be the radius of the circle and $r$ be distance of prticle from the centre of force. Let $\theta$ be the angular coordinate of the particle as shown in the figure.

Therefore using sine rule we get,





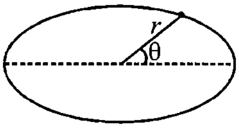
$$\frac{r}{\sin(\pi-2\theta)} = \frac{R}{\sin\theta} \qquad \therefore r = 2R\cos\theta$$

In previous example we have seen that for $r^n = a\cos n\theta$, $F \propto \frac{1}{r^{2n+1}}$. In present case $n = 1$, therefore, force law is $F \propto \frac{1}{r^5}$.

**Polar equation of orbit under** $F = -\frac{k}{r^2}$ **force:** Potential energy corresponding to this force is $V(r) = -\int F dr = -\frac{k}{r}$. Let L be the angular momentum of particle and E be total energy. Therefore,

$L = mr^2\dot{\theta}$ and $E = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{k}{r}$

Differential equation of orbit is $\frac{\partial^2 u}{\partial\theta^2} + u = \frac{-m f(r)}{L^2u^2}$



Using $f(r) = -\frac{k}{r^2} = -ku^2$ we get $\frac{\partial^2 u}{\partial\theta^2} + u = +\frac{mk}{L^2}$

Solution of above differential equation gives $\boxed{r = \frac{L^2/mk}{1 + \sqrt{1 + 2E.\frac{L^2}{mk^2}}\cos\theta}}$

This is polar equation of orbit under $F = -\frac{k}{r^2}$ force. The above is rewritten as,

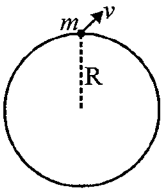
$\boxed{r = \frac{l}{1 + e\cos\theta}}$ where $l = \frac{L^2}{mk}$ = Latus rectum of the conic section (curve) represented by above equation and $e = \sqrt{1 + 2E\frac{L^2}{mk^2}}$ is eccentricity of the orbit.

**Classification of orbits:** From polar equation of orbit we see that under a given force law orbit of a particle depends on its energy. Therefore, in present case,

| **Energy** | **Eccentricity** | **Orbit** |
|---|---|---|
| $\frac{-mk^2}{2L^2} < E < 0$ | e < 1 | Elliptical |
| $E = 0$ | e = 1 | Parabola |
| $E > 0$ | e > 1 | Hyperbola |
| $E = \frac{-mk^2}{2L^2}$ | e = 0 | Circle |

Therefore if nature of orbit is to be found for $F = -\frac{k}{r^2}$ we can just check if its energy satisfies any of the above energy conditions.

**Example:** A particle is thrown from earth's surface with speed $v=\sqrt{\frac{1.5GM}{R}}$, where $M$ and $R$ are mass and radius of the earth respectively. What is the nature of the path of the particle.



**Soln.** Force on the particle is $F=\frac{-GMm}{r^2}$

Therefore potential energy of the particle is, $V=-\int F dr=\frac{-GMm}{r}$

Near earth's surface $V=\frac{-GMm}{R}$

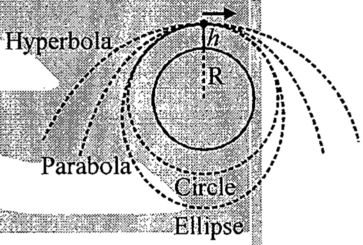
Total energy of the particle is $E=\frac{1}{2}mv^2-\frac{GMm}{R}=\frac{1}{2}m.\left(\frac{1.5GM}{R}\right)-\frac{GMm}{R}=\frac{-GMm}{4R}<0$

Since $E<0$, therefore path of the particle will be elliptical.

**Nature of orbit from velocity of projectile:** In the previous example we have seen that the orbit of projectile depends on the initial velocity with which it is thrown. In the following we give nature of orbit for given initial velocity.

Suppose a particle is thrown from some height above the earth's surface as shown in figure.

| **Velocity** | **Path** |
|---|---|
| $\sqrt{\frac{GM}{R+h}}$ | Circle |
| $\sqrt{\frac{GM}{R+h}}<\text{speed}<\sqrt{\frac{2GM}{R+h}}$ | Ellipse |
| $\sqrt{\frac{2GM}{R+h}}$ | Parabola |
| Speed $>\sqrt{\frac{2GM}{R+h}}$ | Hyperbola |



**Example:** For a particle moving under gravitational force pericentre distance in parabolic orbit is $r_p$ while the radius of the circular orbit with same angular momentum is $r_c$. What is relation between $r_p$ and $r_c$?

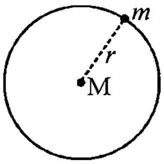
**Soln.** Pericentre distance is the minimum distance $(\cos\theta=\max=1)$ and for parabolic orbit $e=1$.

Therefore, $r_{\min}=\frac{l}{1+e\cos\theta}=\frac{l}{2}=r_p$

for circular orbit $e=0$, therefore $r_c=\frac{l}{1+e\cos\theta}=l\quad\therefore r_p=\frac{r_c}{2}$

**Energy and angular momentum of planet:** Let $m$ be the mass of planet and M be the mass of the sun. Let us assume that sun is stationary and planet revolves around the sun. Here we will use polar equation of orbit to get the expression.

**Circular orbit:** Let the orbit is circular and its radius is $r$, eccentricity of circular orbit is zero.



$$\therefore \quad e=\sqrt{1+2E.\frac{L^2}{k^2}}=0$$

$$\therefore \quad E=\frac{-mk^2}{2L^2} \qquad \text{... (i)}$$

From polar equation of orbit $r=\dfrac{L^2/mk}{1+0}$

$$\therefore \quad L^2=mkr \qquad \text{... (ii)}$$

for gravitational force $k=GMm$, $L=\sqrt{Gm^2Mr}$

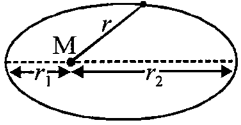
from (i), $E=\dfrac{-m(GMm)^2}{2.Gm^2Mr}=\dfrac{-GMm}{2r}$

if $v$ be the orbital speed then $L=mvr \Rightarrow v=\dfrac{L}{mr}=\sqrt{\dfrac{GM}{r}}$

Therefore, for the planet in circular orbit angular momentum energy and speed are given as

and $\boxed{L=m\sqrt{GMr},\; E=-\dfrac{GMm}{2r},\; v=\sqrt{\dfrac{GM}{r}}}$

**Elliptical orbit:** Let $r_1$ and $r_2$ be minimum and maximum distance of planet from the sun



From polar equation of orbit $r=\dfrac{L^2/mk}{1+\sqrt{1+\dfrac{2E.L^2}{mk^2}}\cos\theta}$

for minimum distance $\cos\theta=1$

$$\therefore \quad r_1=\frac{L^2/mk}{1+\sqrt{1+\dfrac{2E.L^2}{mk^2}}} \quad \text{or} \quad 1+\sqrt{1+\frac{2E.L^2}{mk^2}}=\frac{L^2}{mkr_1} \qquad \text{... (a)}$$

Similarly for maximum distance $(\cos\theta=-1)$

$$1-\sqrt{1+\frac{2E.L^2}{mk^2}}=\frac{L^2}{mkr_2} \qquad \text{... (b)}$$

Add equations (a) and (b) and use $k=GMm$ to get $L=m\sqrt{\dfrac{2GMr_1.r_2}{r_1+r_2}}$

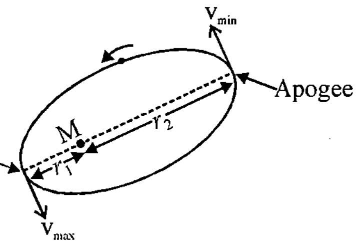
Put value of L in any of the two equations to get $E=-\dfrac{GMm}{r_1+r_2}$

Therefore, for a planet in elliptical orbit energy and angular momentum are given as

$$\boxed{E=-\frac{GMm}{r_1+r_2},\; L=m\sqrt{\frac{2GMr_1.r_2}{r_1+r_2}}}$$

**Maximum and minimum velocity in elliptical orbit:** Angular momentum of planet is conserved therefore its velocity is maximum at perigee and minimum at apogee.

Therefore, $m v_{\max} . r_{\min} = L$ or $v_{\max} = \dfrac{L}{m r_{\min}} = \dfrac{L}{m r_1}$

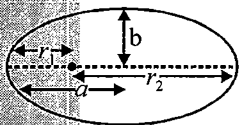


Using value $L = m\sqrt{\dfrac{2GMr_1.r_2}{r_1+r_2}}$ we get $v_{\max} = \sqrt{\dfrac{2GM\,r_2}{r_1(r_1+r_2)}}$

Similarly $m v_{\min} . r_{\max} = L$ gives $v_{\min} = \dfrac{L}{mr_2} = \sqrt{\dfrac{2GMr_1}{r_2(r_1+r_2)}}$

Therefore, $\boxed{v_{\max} = \sqrt{\dfrac{2GMr_2}{r_1(r_1+r_2)}},\ v_{\min} = \sqrt{\dfrac{2GMr_1}{r_2(r_1+r_2)}}}$

**E, L, $v_{max}$, $v_{min}$ in terms of ellipse parameters:** Let $a$, $b$ and $e$ be the semi-major axis, semi-major axis and eccentricity of ellipse.



$\therefore\ r_1 = a(1-e),\ r_2 = a(1+e),\ e = \sqrt{1-\dfrac{b^2}{a^2}}$

Using result of previous section we get

$E = \dfrac{-GMm}{2a}$ (E depends only on length of major axis)

$L = m\sqrt{GMa(1-e^2)}$, $v_{\max} = \sqrt{\dfrac{GM}{a}\left(\dfrac{1+e}{1-e}\right)}$, $v_{\min} = \sqrt{\dfrac{GM}{a}\left(\dfrac{1-e}{1+e}\right)}$

**Example:** Ratio of maximum to minimum speed of a planet revolving around the sun in an elliptical orbit is 2:1, What is eccentricity of the orbit?

**Soln.** Given $\dfrac{v_{\max}}{v_{\min}} = \dfrac{2}{1}$ $\therefore \dfrac{\sqrt{\dfrac{GM}{a}\left(\dfrac{1+e}{1-e}\right)}}{\sqrt{\dfrac{GM}{a}\left(\dfrac{1-e}{1+e}\right)}} = \dfrac{2}{1}$ or $\dfrac{1+e}{1-e} = \dfrac{2}{1}$ $\therefore e = \dfrac{1}{3}$

**Example:** An object is thrown from the earth's surface with initial speed $\sqrt{\dfrac{1.5GM}{R}}$ due to which it moves in an elliptical orbit. What is the semi-major axis of the ellipse.

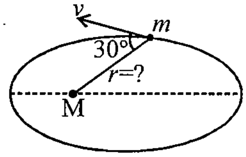
**Soln.** Total energy of the particle

$$E = \frac{-GMm}{R} + \frac{1}{2}mv^2 = \frac{-GMm}{R} + \frac{1}{2}m.\frac{1.5GM}{R} = \frac{-GMm}{4R}$$

In central force motion energy of the particle is conserved. Therefore its energy in elliptical orbit will have the same value. Let $a$ be semi-major axis of the ellipse.

Therefore, $E = \dfrac{-GMm}{2a} = \dfrac{-GMm}{4R}$ $\therefore a = 2R$

**Example:** A planet is revolving around the sun in elliptical orbit of semi-major axis '$a$' and eccentricity $\sqrt{3}/2$. At certain instant of time velocity vector of planet makes 30° with the radial line. What is distance of planet from the sun at this instant.



**Soln.** Energy of the planet is

$$E=\frac{-GMm}{r}+\frac{1}{2}mv^2 \qquad \text{... (i)}$$

Angular momentum of the planet is

$$L=mvr\sin 30°=\frac{mvr}{2} \qquad \text{... (ii)}$$

We know that $L=m\sqrt{GMa(1-e^2)},\ e=\frac{\sqrt{3}}{2}$

$$\therefore \quad \frac{mvr}{2}=m\sqrt{GMa\frac{1}{4}} \quad \text{or} \quad v=\frac{\sqrt{GMa}}{r}$$

We know that $E=\frac{-GMm}{2a}$.

Therefore from (i) we get, $\frac{-GMm}{2a}=\frac{-GMm}{r}+\frac{1}{2}m.\frac{GMa}{r^2}$

or $\quad -\frac{1}{2a}=-\frac{1}{r}+\frac{a}{2r^2} \quad$ or $\quad r^2-2ar+a^2=0 \quad \therefore r=a$

**Example:** A planet is revolving around the sun in a circular orbit. Due to some reason the speed of the planet suddenly becomes double. What is new orbit of the planet.

**Soln.** Orbital speed of the planet is $\sqrt{\frac{GM}{r}}$

New speed of the planet $=2\sqrt{\frac{GM}{r}}$

Therefore, new energy of the planet $=\frac{1}{2}mv^2-\frac{GMm}{r}=\frac{1}{2}m.\frac{4GM}{r}-\frac{GMm}{r}=\frac{2GMm}{r}>0$

Total energy of the planet becomes positive on doubling its speed therefore new orbit of the planet will be hyperbolic.

**Effective potential (Reducing 2-d problem into 1-d problem):** Under central force, a particle has planar motion. That is motion of the particle is 2-dimensional. Therefore total energy is written as $E=\frac{1}{2}mv^2+V(r)$

or, $\quad E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+V(r)$

Angular momentum of the particle is constant of motion and it is given as, $L=mr^2\dot{\theta}$

$$\therefore \quad \dot{\theta}=\frac{L}{mr^2}$$

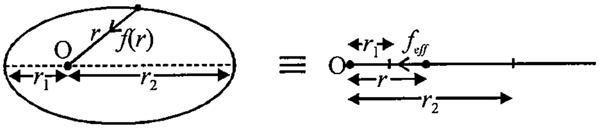
Let us put the value of $\dot\theta$ in the energy expression to get $E=\frac{1}{2}m\dot r^2+\frac{L^2}{2mr^2}+V(r)$

or $$\boxed{E=\frac{1}{2}m\dot r^2+V_{eff}(r)} \quad \text{... (a)}$$

where $$\boxed{V_{eff}(r)=\frac{L^2}{2mr^2}+V(r)}$$

Equation (a) can be interpreted as follows.

Two dimensional motion under a central potential $V(r)$ is equivalent to radial motion under effective potential $V_{eff}(r)$.

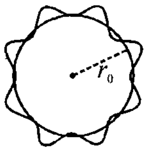


$$f=\frac{-\partial V}{\partial r} \qquad f_{eff}=\frac{-\partial V_{eff}}{\partial r}$$

Therefore, reason for defining an effective potential is that it is easier to discuss 1-d problem than 2-d problem. Before we start dealing with problem let us first understand how the interpretation in one dimension is different from that in two dimension.

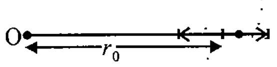
**Actual 2-D problem**

- Energy $E=\frac{1}{2}mv^2+V(r)$
- At turning points velocity is ⊥ to the radial line
- Circular motion under a central force is uniform motion and $f(r)=\frac{mv^2}{r}$
- Oscillation about stable radius is shown as



**Equivalent 1-D problem**

- Energy $E=\frac{1}{2}m\dot r^2+V_{eff}(r)$
- At turning points $\dot r=0$. Therefore it appears as if particle stops there.
- In 1-d description of circular motion particle appears stationary, $\dot r=0$ and $f_{eff}=0$.
- An equivalent oscillation in 1–$d$ is linear oscillation



**Finding the nature of possible orbits under a given central force / potential:** In previous section we have seen that various types of orbits are possible under $F=-k/r^2$ force and the nature of orbit depends on energy of particle. It is tedious task to find polar equation of orbit for various forces and then conclude about the nature of orbit for a given energy as we did in previous section. Therefore instead of discussing the problem in 2-d we discuss equivalent problem in 1-d.

In equivalent 1-d problem total energy is written as $E=\frac{1}{2}m\dot r^2+V_{eff}$

or $\frac{1}{2}m\dot r^2=E-V_{eff}$

Therefore, $E$ must be greater or equal to $V_{eff}$. Thus we conclude that if we plot $V_{eff}$ versus $r$, movement of particle will be limited to that region only where $V_{eff}$ line lies below the total energy line. That is particle cannot go in the region where $V_{eff}$ line is above the total energy (E) line.

**Dynamics under Inverse square force $F = -k/r^2$ or $V(r) = -k/r$:**

Let $m$ be mass of particle and L be its angular momentum.

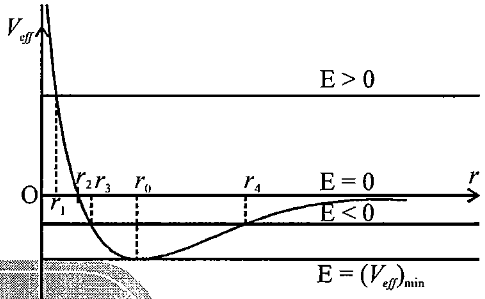
Therefore, $V_{eff}(r) = V(r) + \dfrac{L^2}{2mr^2} = \dfrac{-k}{r} + \dfrac{L^2}{2mr^2}$

The plot of $V_{eff}(r)$ versus $r$ is shown in the figure below. To plot the graph consider the following

If $f(r) = ar^m - br^n$ where $m > n$ then

for $r \to 0$ lower power term ($br^n$) dominates

for $r \to \infty$ higher power term ($ar^m$) dominates



From the plot of $V_{eff}$ versus $r$ we draw following conclusion.

(i) There is a minimum at $r = r_0$. Therefore if particle maintains its distance from centre of force to be equal to $r_0$ then its orbit is stable. i.e. if particle moves in a circle of radius $r_0$ then its orbit is stable.

Therefore to find radius ($r_0$) of stable orbit we use $\left.\dfrac{dV_{eff}}{dr}\right|_{r=r_0} = 0$

(ii) For E > 0 minimum distance of the particle from centre of force is $r_1$ and maximum distance is infinity that is $r_1 \le r \le \infty$. Therefore, for E > 0 orbit is unbounded.

(iii) For E = 0, $r_{min} = r_2$, $r_{max} = \infty$, therefore $r_1 \le r \le \infty$ and orbit is unbounded.

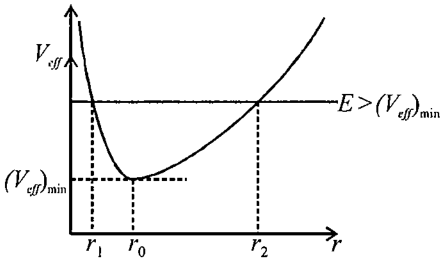
(iv) For E < 0, $r_{min} = r_3$, $r_{max} = r_4$, therefore $r_3 \le r \le r_4$ orbit is bounded.

(v) $E = (V_{eff})_{min}$, only $r = r_0$ is possible, therefore orbit is a circle of radius $r_0$.

(vi) $E < (V_{eff})_{min}$ is not allowed

**Dynamics under Hookes force: $F = -kr$, $V(r) = \frac{1}{2}kr^2$:**

$$V_{eff} = \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$



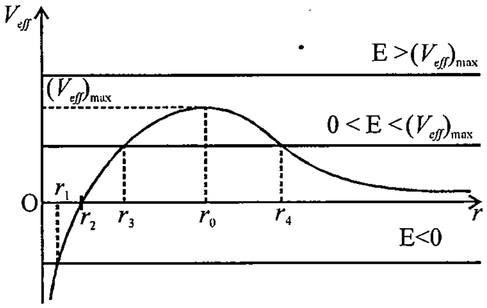
Plot of $V_{eff}$ versus $r$ is shown in the figure and from graph we draw following conclusion.

(i) For $E = (V_{eff})_{min}$ only $r = r_0$ is possible therefore orbit is circular.

(ii) For $E > (V_{eff})_{min}$, $r_{min} = r_1, r_{max} = r_2$ that is $r_1 \le r \le r_2$. Therefore orbit is bounded. Thus we see that under Hooke's force orbit is bounded for all positive energies which are greater than the minimum value of effective potential.

(iii) $E < (V_{eff})_{min}$ is not allowed

**Dynamics under** $F=\frac{-k}{r^4}$ force or $V(r)=\frac{-k}{3r^3}$ :



$$V_{eff}(r)=\frac{-k}{3r^2}+\frac{L^2}{2mr^2}$$

Plot of $V_{eff}(r)$ versus $r$ is shown in the figure.

From the plot we draw following conclusion.

(i) For $E<0$, $r_{min}=0$, $r_{max}=r_1$ that is $0\le r\le r_1$. Therefore motion is bounded.

(ii) For $E=0$, $0\le r\le r_2$. Therefore motion is bounded.

(iii) For $0<E<(V_{eff})_{max}$, $0\le r\le r_3$ or $r_4\le r\le\infty$. Therefore for such value of energy, motion may be bounded or unbounded.

(iv) For $E\ge(V_{eff})_{max}$, $0\le r\le\infty$. Therefore motion is unbounded.

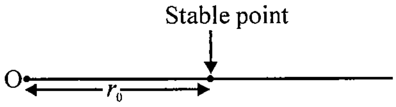
(v) This type of potential has a special feature.

- For $r>r_0$ $\frac{\partial V_{eff}}{\partial r}<0$. Therefore $f_{eff}=\frac{-\partial V_{eff}}{\partial r}>0$ i.e. force on the particle is away from the centre of force.
- For $r<r_0$ $\frac{\partial V_{eff}}{\partial r}>0$. Therefore $f_{eff}=\frac{-\partial V_{eff}}{\partial r}<0$ i.e. force on the particle is towards the centre of force.
- Due to attractive nature of force for $r<r_0$ any object coming closer to the centre of force gets captured if it is able to cross the $r=r_0$ point.

**Example:** A planet of mass $m$ and angular momentum L moves in a circular orbit in a potential, $V(r)=-k/r$ where $k$ is a positive constant find the radius of stable circular orbit. If the planet is slightly perturbed, find angular frequency of radial oscillation.

**Soln.** $V_{eff}=\frac{L^2}{2mr^2}+V(r)=\frac{L^2}{2mr^2}-\frac{k}{r}$

For stable point (orbit) $\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0}=0$

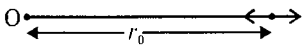


$$\therefore -\frac{L^2}{mr_0^3}+\frac{k}{r_0^2}=0 \text{ or } r_0=\frac{L^2}{mk}$$

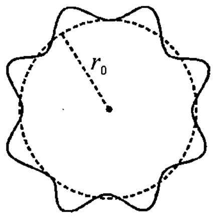
Therefore radius of stable circular orbit is $L^2/mk$

Angular frequency of oscillation about stable point:

$$\omega=\sqrt{\frac{\left.\frac{\partial^2V_{eff}}{\partial r^2}\right|_{r=r_0}}{m}}=\sqrt{\frac{\left(\frac{3L^2}{mr_0^4}-\frac{2k}{r_0^3}\right)}{m}}$$

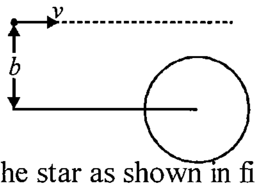


$$= \sqrt{\frac{\frac{1}{r_0^3}\left(\frac{3L^2}{m.\frac{L^2}{mk}}-2k\right)}{m}} = \sqrt{\frac{k}{mr_0^3}} = \sqrt{\frac{k^4m^2}{L^6}} = \frac{mk^2}{L^3}$$



actual radial oscillation is as shown in figure.

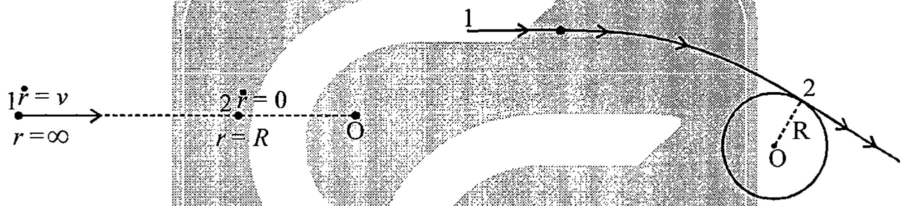
**Example :** A small asteroid is approaching a massive star with a speed $v$ from very large distance, at an impact parameter '$b$' as shown in figure. If the mass of the star is M and its radius is R, then what is the minimum value of $b$ such that the asteroid will miss the star?



**Soln.** Angular momentum of asteroid about centre of star is $L = mvb$

Effective potential energy $V_{eff} = \frac{-GMm}{r} + \frac{L^2}{2mr^2} = \frac{-GMm}{r} + \frac{mv^2b^2}{2r^2}$

For minimum value of $b$ we will have to assume that asteroid just misses the star as shown in figure. Equivalent 1-d problem is also shown in figure



Total energy is conserved in central force motion

Therefore, $E_1 = E_2$

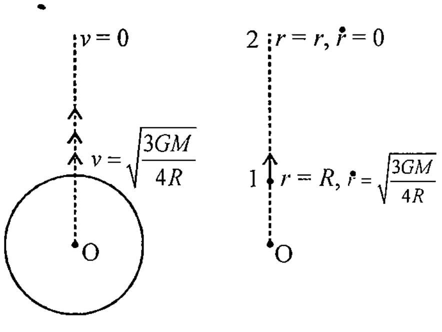
or $\left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{\text{at } r=\infty} = \left(\frac{1}{2}m\dot{r}^2 + V_{eff}\right)_{\text{at } r=R}$

$$\frac{1}{2}mv^2 - \frac{GMm}{\infty} + \frac{mv^2b^2}{\infty} = \left(\frac{1}{2}m0^2 - \frac{GMm}{R} + \frac{mv^2b^2}{2R^2}\right)$$

$\therefore \; v^2 + \frac{2GM}{R} = \frac{v^2b^2}{R^2}$ or $b = R\sqrt{1 + \frac{2GM}{Rv^2}}$

**Example:** A particle is thrown in radially outward direction from earth's surface with initial speed $\sqrt{\frac{3GM}{4R}}$ where M is mass of earth R is radius of earth. Find the height upto which particle goes.

**Soln.** Particle has been thrown radially outward


$v = 0$
$v = \sqrt{\frac{3GM}{4R}}$
O
2 $r = r, \dot{r} = 0$
1 $r = R, \dot{r} = \sqrt{\frac{3GM}{4R}}$
O

therefore $L = 0$, $V_{eff} = V(r) + \frac{L^2}{2mr^2}$

$\therefore \; V_{eff} = \frac{-GMm}{r} + 0$

from conservation of energy we get $E_1 = E_2$

or $\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=R}=\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=r}$

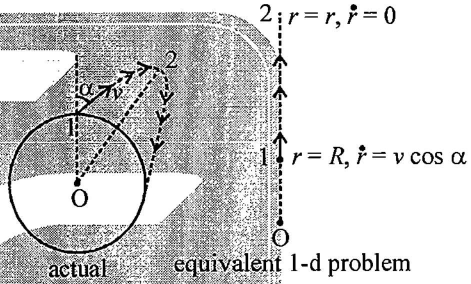
$$\frac{1}{2}m.\frac{3GM}{4R}-\frac{GMm}{R}=0-\frac{GMm}{r}$$

$$\frac{3}{8R}-\frac{1}{R}=-\frac{1}{r} \qquad \therefore\ r=\frac{8R}{5}$$

Therefore, maximum height from earth's surface $= r-R=\frac{3R}{5}$

**Example:** A particle is thrown from the earth's surface with speed $\sqrt{\frac{GM}{R}}$ where $M$ = mass of earth and $R$ = radius of earth. If direction of initial velocity makes an angle $\alpha$ with the outward radial direction, find the maximum distance of the particle from centre of earth.

**Soln.** Angular momentum of the particle is



$$L=mv\sin\alpha R=m\sqrt{\frac{GM}{R}}\sin\alpha\, R$$

$$=m\sqrt{GMR}\sin\alpha$$

$$\therefore\ V_{eff}=V(r)+\frac{L^2}{2mr^2}=\frac{-GMm}{r}+\frac{GMmR}{2r^2}\sin^2\alpha$$

From conservation of energy we get $E_1=E_2$

or $\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=R}=\left(\frac{1}{2}m\dot{r}^2+V_{eff}\right)_{r=r}$ (see the figure)

$$\frac{1}{2}m\frac{GM}{R}\cos^2\alpha-\frac{GMm}{R}+\frac{GMm}{2R}\sin^2\alpha=\frac{-GMm}{r}+\frac{GMmR}{2r^2}\sin^2\alpha$$

Divide both sides by $\frac{GMm}{2R}$ to get $\cos^2\alpha-2+\sin^2\alpha=-2\frac{R}{r}+\frac{R^2}{r^2}\sin^2\alpha$

or $-1=-2\left(\frac{R}{r}\right)+\left(\frac{R}{r}\right)^2\sin^2\alpha \qquad \therefore\ \left(\frac{R}{r}\right)^2\sin^2\alpha-2\left(\frac{R}{r}\right)+1=0$

$$\therefore\ \frac{R}{r}=\frac{+2\pm\sqrt{4-4\sin^2\alpha}}{2\sin^2\alpha} \qquad \frac{R}{r}=\frac{+1\pm\cos\alpha}{\sin^2\alpha}$$

$$\therefore\ r=\frac{R\sin^2\alpha}{1\pm\cos\alpha}=\frac{R(1-\cos^2\alpha)}{(1\pm\cos\alpha)} \qquad \therefore\ r=R(1-\cos\alpha) \quad \text{or} \quad r=R(1+\cos\alpha)$$

Since $r$ must be greater than $r$. Therefore, $r=R(1+\cos\alpha)$

**Condition for stable orbit:** If $V_{eff}$ versus $r$ plot has a minimum then stable circular orbit can exist.

Let the central force is $f(r)=-kr^n$

Therefore, $V_{eff} = V(r) + \frac{L^2}{2mr^2} = -\int f(r)dr + \frac{L^2}{2mr^2} = \frac{kr^{n+1}}{n+1} + \frac{L^2}{2mr^2}$

Condition for existence of minimum is

$$\left.\frac{\partial V_{eff}}{\partial r}\right|_{r=r_0} = 0 \qquad \text{... (i)}$$

and $$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r=r_0} > 0 \qquad \text{... (ii)}$$

using first condition we get $+kr_0^n - \frac{L^2}{mr_0^3} = 0$ or $r_0^{n+3} = \frac{L^2}{mk}$ ... (iii)

using second condition we get, $nkr_0^{n-1} + \frac{3L^2}{mr_0^4} > 0$ or $\frac{k}{r_0^4}\left(nr_0^{n+3} + \frac{3L^2}{mk}\right) > 0$

using equation (iii) we get, $\frac{k}{r_0^4}\left(nr_0^{n+3} + 3r_0^{n+3}\right) > 0$ or $kr_0^{n-1}(n+3) > 0$

Since $k$ and $r_0$ are positive therefore we must have $n+3>0$ or $\boxed{n>-3}$

Thus, if $F = -kr^n$ then $n$ must be greater than $-3$ for orbit to be stable.

**Condition for closed orbit (Bertrand's theorem):** If $F(r) = -kr^n$ then closed orbits are possible only for $n=1$ and $n=-2$ i.e. for $f = -kr$, Hooke's force and for $f(r) = \frac{-k}{r^2}$, inverse square force (gravitation).

**Virial theorem:** If $f \propto r^n$ then average value of potential energy and kinetic energy are related as

$$<K.E> = \frac{n+1}{2} <P.E>$$

for gravitational force $n=-2$ $\quad \therefore \; <K.E> = -\frac{1}{2} <P.E>$

for Hooke's force $n = 1$ $\quad \therefore \; <K.E> = <P.E>$

**Example:** A particle moves under a central potential $V(r) = \frac{-k}{r^m}$. What should be value of $m$ for its orbit to be stable.

**Soln.** Corresponding is $f(r) = -\frac{\partial V}{\partial r} = \frac{-mk}{r^{m+1}} = -mk\,r^{-(m+1)}$

We know that for $f = -kr^n$ condition for stability is $n > -3$.

Therefore for orbit to be stable under given potential we must have.

$-(m+1) > -3$ or $m+1 < 3$ $\therefore m < 2$

**Example:** A particle is moving under a central force $f(r) = \frac{-k}{r^{3-n}}$ for what values of $n$ orbit of the particle can be closed.

**Soln.** We know that for $f(r) = -kr^m$ orbit is closed if $m = -2$ or 1.

Therefore, $-(3-n) = -2 \Rightarrow n = 1$ and $-(3-n) = 1 \Rightarrow n = 4$.

Therefore, values of $n$ for which orbit can be closed are 1 and 4.

**Keppler's law of planetary motion:** Keppler had given three empirical laws of planetary motion on the basis of analysis of astronomical data collected by his master Tycho Brahe.

**First Law:** All the planets move in elliptical orbit around the sun with sun being at one focus of the ellipse.

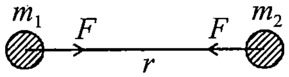
**Second Law:** Line joining the centre of planet to the centre of the sun sweeps equal area in equal interval of time. That is areal velocity of a planet is constant.

**Third Law:** Square of time of revolution is proportional to cube of semi-major axis of elliptical orbit.

i.e. $T^2 \propto a^3$

***Newton's law of gravitation*** :

It states that every matter particle attracts every other matter particle with a force which is proportional to product of masses and inversely proportional to square of distance between them.



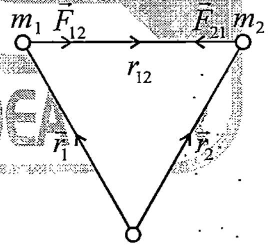
$$F = \frac{G m_1 \cdot m_2}{r^2}$$

$G$ is universal gravitational constant. Its value is $6.67 \times 10^{-11}\ N m^2 / Kg^2$

The above expression for gravitational force is strictly valid for point masses and for two spherically symmetric masses.

If $\vec{r}_1$ and $\vec{r}_2$ be position vectors of $m_1$ and $m_2$ then $r = |\vec{r}_1 - \vec{r}_2|$ Therefore,

$$F = \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^2}$$



or, vector form of force on $m_1$ due to $m_2$ is

$$\vec{F}_{12} = \frac{G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|} \hat{r}_{12} = \frac{G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|^2} \cdot \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\vec{F}_{12} = \frac{-G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_1 - \vec{r}_2)$$

and $$\vec{F}_{21} = \frac{-G m_1 \cdot m_2}{|\vec{r}_1 - \vec{r}_2|^3} (\vec{r}_2 - \vec{r}_1)$$

**Note:** All matter particles attracts each other due to gravitational force however if an air bubble and a particle is lying inside water they appear to repel each other.

Gravitational field: Gravitational force per unit test mass is called gravitational field $(\vec{E})$

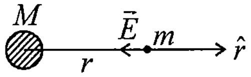
$$\boxed{\vec{E} = \frac{\vec{F}}{m}}$$ unit :N/Kg

**Gravitational field due to a point mass:**

Let $M$ be mass of a point particle and at a distance $r$ form it we have to find gravitational field.

Let us place a test mass there.

$$\therefore \qquad E = \frac{F}{m} = \frac{\frac{GMm}{r^2}}{m}$$



$$E = \frac{GM}{r^2}$$

or $$\boxed{\vec{E} = -\frac{GM}{r^2}\hat{r}}$$

$\vec{E}$ is always directed towards the particle.

From Newton's 2nd law $\vec{F}/m = \vec{a}$. Therefore gravitational field is equal to acceleration of test mass due to gravitational force i.e. gravitational field = acceleration due to gravity.

**Gauss law for garvitational field:**

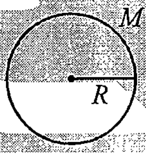
$$\boxed{\oint \vec{E}\cdot d\vec{s} = -4\pi\, G\,\mathrm{Men}_c}$$

Total flux of gravitational field through a closed surface is equal to $-4\pi G$ times the total mass enclosed by the surface.

Its applicable in all cases. However it cab be used to calculate gravitational field only in the cases involving spherical, cylindrical or planar symmetry.

In gravitation we get result similar to electrostatic results. We can easily obtain gravitational results from electrostatic results and vice versa if we make following changes.

$$\frac{1}{4\pi E_0} \leftrightarrow G, \; Q \leftrightarrow M$$



Some important results in gravitational

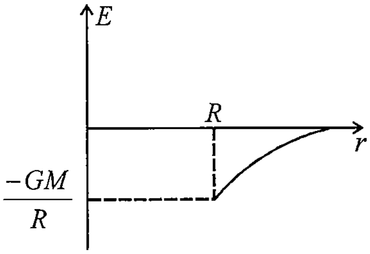
Uniform thin hollow shell (mass = $M$, radius = $R$)

Field inside is zero

i.e. $E = 0$ for $r < R$

Field outside is inversely proportional to $r^2$

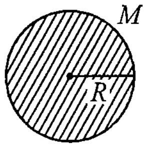
i.e. $\quad E = \frac{-GM}{r^2} \quad r > R$



Uniform solid sphere (mass = $M$, radius = $R$)

Field inside is proportional to distance from center

i.e. $\quad E = \frac{-GM}{R^3} r \;$ for $\; r < R$



Field outside is inversely proportional to $r^2$

i.e. $\quad E = \frac{-GM}{r^2} \; r > R$

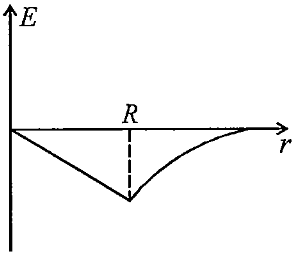
**Acceleration to gravity near Earth:**

Since acceleration due to garvity is equal to gravitational field, therefore considering earth as a uniform solid sphere we get

$$g=\frac{GM}{R^2} \text{ on earth's surface}$$

$$g=\frac{GM}{r^2}=\frac{GM}{(R+h)^2} \text{ at '}h\text{' height above earth's surface}$$

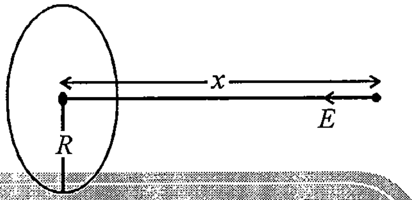
$$g=\frac{GMr}{R^3}=\frac{GM}{R^3}(R-d) \text{ at '}d\text{' depth below earth's surface}$$



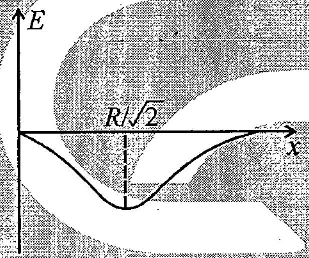
**Uniform thin ring**

Gravitational field at an axial point

$$E=\frac{-GMx}{(R^2+x^2)^{3/2}}$$



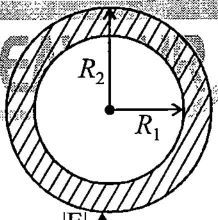
Field is maximum at a distance $x=\frac{R}{\sqrt{2}}$ from center



**Example:** A thick shell of inner and outer radius $R_1$ and $R_2$ has uniform mass density $P$. Calculate gravitational field and plot it verses radius distance.

**Soln.** Gravitational field due to spherically symmetric change distribution at a distance $r$ from center is given as

$$E=\frac{-GMen_c}{r^2}$$

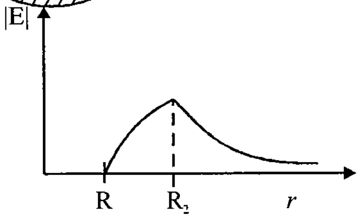


Where Men$_c$ is mass enclosed by sphere of radius $r$.

Therefore, for $r<R$, Men$_c = 0$

for $R_1<r<R_2$ Men$_c = \rho\frac{4}{3}\pi(r^3-R_1^3)$

for $r>R_2$ Men$_c = \rho\frac{4}{3}\pi(R_2^3-R_1^3)$



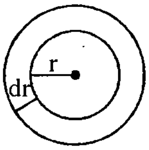
Thus, $E=0$ for $r<R_1$

$$E=\frac{-4\pi\rho G}{3}\left(r-\frac{R_1^3}{r}\right) \text{ for } R_1<r\ R_2$$

$$E=\frac{-4\pi\rho G}{3}\left(\frac{R_2^3-R_1^3}{r^2}\right) \text{ for } r>R_2$$

radial variation of $E$ is shown above.

**Gravitational self energy:**
Solid sphere,



$$\rho_m = \frac{M}{\frac{4}{3}\pi R^3}$$

Mass inside the sphere of radius 'r' = $\rho_m\left(\frac{4}{3}\pi r^3\right) = M\left(\frac{r^3}{R^3}\right)$

dm = Mass in the ring of rod but r and r + dr = $\rho_m 4\pi r^2 dr = \frac{3Mr^2 dr}{R^3}$

Work done bringing dm from $\infty$ to r is:

$$dV = -\frac{Gdm}{r}\left(\frac{Mr^3}{R^3}\right) = -G\frac{Mr^3}{rR^3}\frac{3Mr^2}{R^3}dr = -\left(G\frac{3M^2}{R^6}\right)rdr$$

$$U = -\int_0^R \frac{G3M^2}{R^6}r^4 dr = -\frac{3GM^2}{R}\frac{R^5}{5} = \boxed{-\frac{3}{5}\frac{GM^2}{R}}$$

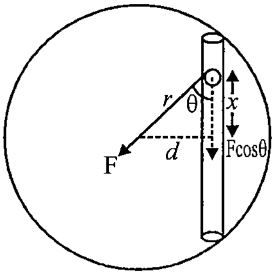
## SOLVED PROBLEMS

1. An object of mass '*m*' is dropped in a tunnel dug across a chord of earth. If *d* be the distance of the centre of chord from the earth's centre, calculate the period of oscillation of the object.

**Soln.** The situation is shown in the figure.

Let *r* be the distance of the object from the centre of the sphere at certain instant of time.

$\therefore$ Force on the object



$$F = \frac{GMm}{R^3}\cdot r$$

$\therefore$ Restoring force towards the centre of the tunnel = $F\cos\theta$

$$= F\cdot\frac{x}{r} \qquad = \frac{GMm}{R^3}\cdot x$$

$\therefore$ acceleration of object along the tunnel

$$\vec{a} = \frac{\vec{F}_{\text{restoring}}}{m} \qquad \vec{a} = \frac{GM}{R^3}(-\vec{x})$$

or $\vec{a} = \left(\frac{GM}{R^2}\right)\cdot\frac{1}{R}(-\vec{x})$

since $g = \frac{GM}{R^2}$ (acceleration due to gravity)

$\therefore\ \vec{a} = \left(\frac{g}{R}\right)(-\vec{x})$

comparing with equation of S.H.M. we get

$\omega^2 = g/R \Rightarrow T = 2\pi\sqrt{R/g}$

2. Show that the vector, $\vec{A} = \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r}$

for planetary system is a constant of motion where $\vec{r}$ is position vector, $\vec{p}$ is momentum and $\vec{L}$ is angular momentum.

**Soln.** We have to show that $\frac{d\vec{A}}{dt} = 0$

$$\frac{d\vec{A}}{dt} = \frac{d\vec{p}}{dt} \times \vec{L} + \vec{p} \times \frac{d\vec{L}}{dt} - \frac{mk}{r}\frac{d\vec{r}}{dt} - mk\vec{r}\frac{d(1/r)}{dt}$$

Here $\frac{d\vec{p}}{dt} = \vec{F} = -\frac{k}{r^2}\hat{r}$

Force is radial, therefore, $\frac{d\vec{L}}{dt} = 0$

$$\therefore \quad \frac{d\vec{A}}{dt} = -\frac{k}{r^2}\hat{r} \times (\vec{r} \times \vec{p}) - \frac{mk}{r}\frac{d\vec{r}}{dt} + \frac{mk}{r^2}\vec{r}\frac{dr}{dt}$$

$$= \frac{-k\vec{r}}{r^2}\hat{r} \cdot \vec{p} + \frac{k}{r^2}\vec{p}(\vec{r} \cdot \hat{r}) - \frac{mk}{r} \cdot \frac{d\vec{r}}{dt} + \frac{mk}{r^2}\vec{r}\frac{dr}{dt}$$

As, $\hat{r} \cdot \vec{p} = \frac{mdr}{dt}$ and $\vec{r} \cdot \hat{r} = r$ and $\vec{p} = m\frac{d\vec{r}}{dt}$

$\therefore$ first and fourth terms and second and third terms cancel out.

$$\therefore \quad \frac{d\vec{A}}{dt} = 0 \Rightarrow \vec{A} = \text{constant}$$

3. A satellite is moving around earth in a circular orbit of radius $R$. The time period $T$ of the satellite is

(a) proportional to $R$ (b) proportional to $R^2$ **[GP-2005]**

(c) proportional to $R^{3/2}$ (d) independent of $R$

**Soln.** (c) According to Kepler's third law $T \propto R^{3/2}$

4. The escape velocity from the earth is $v_0$. For a planet with radius three times and density twice the earth the escape velocity will be **[GP-2007]**

(a) $v_0\sqrt{2}$ (b) $3v_0\sqrt{2}$ (c) $2v_0\sqrt{2}$ (d) $v_0\sqrt{6}$

**Soln.** (b)

Escape velocity from a planet of radius $R$ and mass $M$ is given as

$$v_{\text{escape}} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2G\frac{4}{3}\pi R^3\rho}{R}} \qquad \therefore \quad \frac{v_1}{v_2} = \sqrt{\frac{R_1^2\rho_1}{R_2^2\rho_2}}$$

If $v_1 = v_0$ and $R_2 = 3R_1$, $\rho_2 = 2\rho_1$

$$\text{then } \frac{v_0}{v_2} = \sqrt{\frac{1}{9 \times 2}} \qquad \therefore \quad v_2 = 3v_0\sqrt{2}$$

5. A satellite moves in elliptical orbit around the earth. The minimum and the maximum distances of satellite from the earth are $6.3 \times 10^5$m and $3.63 \times 10^6$m, respectively. The radius of earth is $6.37 \times 10^6$m. The ratio of speed of the satellite at apogee at its speed at perigee is **[GP-2012]**

(a) 0.3 (b) 0.35 (c) 0.6 (d) 0.7

**Soln.** (d) Maximum distance of satellite from earth's centre $= 3.63 \times 10^6 + 6.37 \times 10^6$
$= 10.00 \times 10^6$m

Minimum distance of satellite from earth's centre $= 0.63 \times 10^6 + 6.37 \times 10^6$
$= 7.00 \times 10^6$m

$$\therefore \quad \frac{v_{apogee}}{v_{perigee}} = \frac{\text{minimum distance}}{\text{maximum distance}} = \frac{7.00\times10^6}{10.00\times10^6} = 0.7$$

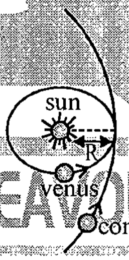
6. Suppose a comet of mass $\sqrt{20}\times10^{13}$ kg is moving in a parabolic orbit around the sun and has a possibility of colliding at its perihelion with venus. Assuming that venus is moving in a circular orbit around the sun and takes 225 days to complete a revolution around the sun. Calculate the angular momentum of comet about the sun. Neglect the interaction between the comet and venus. (Mass of sun = $2\times10^{30}$kg, G = $\frac{20}{3}\times10^{-11}$

5 unit, $\pi^2 = 10$) **[GP-2010]**

**Soln.** For venus (Mass = $M_v$)

$$\frac{GM_vM_s}{R^2} = \frac{M_vv_0^2}{R}, v_0 = \text{ orbital speed}$$

$$v_0 = \sqrt{\frac{GM_s}{R}}$$



$$\therefore \quad T = \frac{2\pi R}{v_0} = 2\pi\sqrt{\frac{R^3}{GM_s}}$$

$$\therefore \quad R^3 = \frac{T^2GM_s}{4\pi^2}$$

For comet (Mass = $M_c$)

Total energy = 0 ($\because$ orbit is parabolic)

$$\therefore \quad \frac{1}{2}M_cv_p^2 - \frac{GM_cM_s}{R} = 0, \qquad v_p = \text{velocity at perihelion}$$

$$\therefore \quad v_p = \sqrt{\frac{2GM_s}{R}}$$

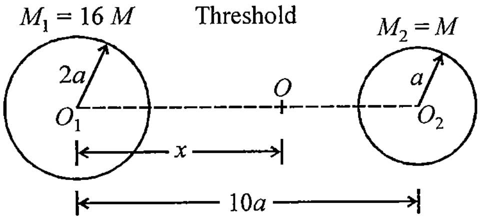
$\therefore$ Angular momentum of comet is

$$L = M_cv_pR = M_c\sqrt{\frac{2GM_s}{R}}R = \sqrt{2GM_sM_c^2R}$$

$$= \sqrt{2GM_sM_c^2}\cdot\left[\frac{T^2GM_s}{4\pi^2}\right]^{1/6}$$

7. Distance between the centers of two stars is $10a$. The masses of these stars are $M$ and $16\,M$ and their radii $a$ and $2a$ respectively. A body of mass $m$ is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star ? Obtain the expression in terms of $G$, $M$ and $a$.

**Soln.** The situation is shown in figure.



Let O be the point along $O_1O_2$ where gravitational intensities due to both the stars balance each other. Let $O_1O = x \cdot x$ is in $a$ units.

Now, $-\dfrac{GM_1}{x^2} = -\dfrac{GM_2}{(10-x)^2}$ or $16(10-x)^2 = x^2$ or $x = 8a$.

Potential energy of the body on surface of larger star

$$U_1 = -\frac{Gm(16M)}{2a} - \frac{GmM}{8a} = -\frac{65GMm}{8a}$$

$$U_{\text{Threshold}} = -\frac{GMm}{2a} - \frac{G(16M)m}{8a} = -\frac{5GMm}{2a}$$

Now, $U_1 + \dfrac{1}{2}mv_{\min}^2 = U_{\text{Threshold}}$ or $-\dfrac{65GMm}{8a} + \dfrac{1}{2}mv_{\min}^2 = -\dfrac{5GMm}{2a}$

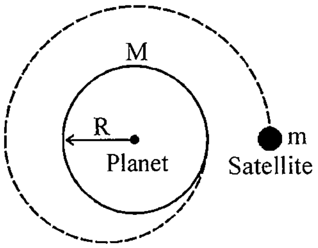
or $\dfrac{1}{2}mv_{\min}^2 = \dfrac{65GMm}{8a} - \dfrac{5GMm}{a} = \dfrac{45}{8}\dfrac{GMm}{a}$

$$v_{\min}^2 = \frac{45}{4}\frac{GM}{a} = \frac{9\times 5}{4}\times\frac{GM}{a}$$

$$v_{\min} = \frac{3}{2}\sqrt{\left(\frac{5GM}{a}\right)}.$$

8. An artificial satellite of mass $m$ of a planet of mass $M$, revolves in a circular orbit whose radius is $n$ times the radius $R$ of the planet. In the process of motion, the satellite experiences a slight resistance due to cosmic dust. Assuming resistance force on satellite to depend on velocity as $F = av^2$ where a is constant. Calculate how long the satellite will stay in orbit before its falls onto the planet's surface.

**Soln.** The situation is shown in figure.



Let an instant, the distance of the satellite from the center of the planet be $x$.

Velocity of satellite, $v = \sqrt{\left(\frac{GM}{x}\right)}$ ... (i)

Given that resistive force $F = av^2$

Therefore, $F = a\left(\frac{GM}{x}\right)$ ... (ii)

During a very small time interval $dt$, distance travelled by the satellite is

$$v\,dt = \sqrt{\left(\frac{GM}{x}\right)}\,dt \qquad \text{... (iii)}$$

So, the workdone by satellite against resistance of cosmic dust

$$dW = F\cdot(v\,dt) = a\sqrt{\left(\frac{GM}{x}\right)}\times\sqrt{\left(\frac{GM}{x}\right)}\,dt$$

or $$dW = a\left(\frac{GM}{x}\right)^{3/2} dt \qquad \text{... (iv)}$$

Due to this work done, the energy of satellite decreases by the same amount and consequently the radius $x$ also decreases simultaneously.

We know that $E = -\frac{1}{2}\frac{GMm}{x}$ or $dE = \frac{1}{2}\frac{GMm}{x^2}dx$ ... (v)

But $dE = -dW$

$$\therefore \quad \frac{1}{2}\frac{GmM}{x^2}dx = -a\left(\frac{GM}{x}\right)^{3/2} dt$$

$$dt = -\frac{m}{2a\sqrt{GM}}\times\frac{dx}{\sqrt{x}} \qquad \text{... (vi)}$$

Integrating equation (vi), we get

$$\int_{t=0}^{t=t} dt = -\frac{m}{2a\sqrt{GM}}\int_{x=nR}^{x=R}\frac{dx}{\sqrt{x}} \qquad (\because \text{ when } t=0,\ x=nR \text{ and at } t=t,\ x=R)$$

Solving we get, $t = \frac{m\sqrt{R}\left(\sqrt{n}-1\right)}{a\sqrt{(GM)}}$.

# Chapter 9

# Rigid Body Dynamics

A rigid body is collection of large number of particles. Distance between each pair of particle remains constant. i.e. $|\vec{r}_i - \vec{r}_j|$ = constant. Rotational dynamics of rigid body involves basically dynamics of large number of particles. Therefore it is not possible to describe the dynamics in terms of movement of each particle. However, by introducing concept of moment of inertia or inertia tensor rotational dynamics of a rigid body can be systematically discussed. Moment of inertia or inertia tensor collectively takes into account the movement of each particle.

**Moment of inertia:** Moment of inertia of a system of particles about an axis of rotation is defined as

$I = \sum_i m_i r_i^2$ (for discrete case)

where $m_i$ = mass of each point particle

$r_i$ = perpendicular distance from the axis of rotation

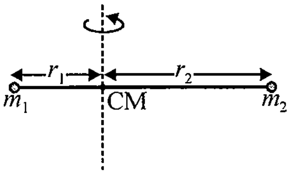
or $I = \int r_\perp^2 dm$ (for continuous case)

where $dm$ = elementary mass

$r_\perp$ = perpendicular distance of elementary mass from the axis of rotation.

**Example:** Two particles of masses $m_1$ and $m_2$ are kept '$r$' distance apart. Calculate moment of inertial of the system about an axis passing through the centre of mass and perpendicular to the line joining the masses.

**Soln.** Let $m_1 > m_2$, therefore centre of mass of the system lies nearer to $m_1$.

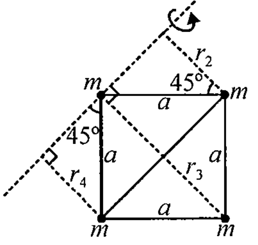


$\therefore \quad I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$

Since, $r_1 = \dfrac{m_2 r}{m_1 + m_2}$ and $r_2 = \dfrac{m_1 r}{m_1 + m_2}$

$$\therefore \; I = m.\left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\frac{mr}{m_1 + m_2}\right)^2 = \frac{m_1 . m_2}{m_1 + m_2} r^2$$

**Example:** Four particles each of mass '$m$' are kept at the corners of a square of side '$a$'. Calculate moment of inertia about an axis passing through one corner. The axis is parallel to diagonal of square.



**Soln.** $I = \sum_i m_i r_i^2 = m(r_1^2 + r_2^2 + r_3^2 + r_4^2)$

from figure shown, $r_1 = 0, r_2 = a\cos 45°, r_3 = a\sqrt{2}, r_4 = a\sin 45°$

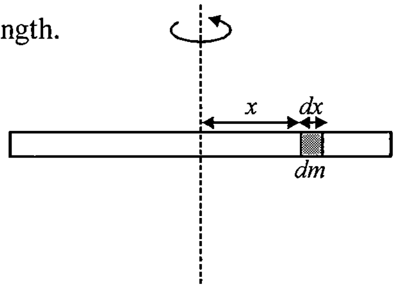
$$\therefore \; I = m\left[0 + \left(\frac{a}{\sqrt{2}}\right)^2 + (a\sqrt{2})^2 + \left(\frac{a}{\sqrt{2}}\right)^2\right] = 3ma^2$$

**Moment of inertia of a uniform thin rod:** Let M = mass of rod and L = length of the rod.

Moment of inertia of a continuous mass distribution is defined as $I=\int r_{\perp}^{2}dm$

Let $m$ take an element of width $dx$ at a distance $x$ from the axis of rotation. Therefore $dm=\frac{M}{L}dx$

**Case 1:** About an axis passing through its centre and $\perp$ to its length.



Here, $r_{\perp}=x$

$$\therefore\quad I=\int_{-L/2}^{+L/2}x^{2}\frac{M}{L}dx=\frac{M}{L}\int_{-L/2}^{L/2}x^{2}dx$$

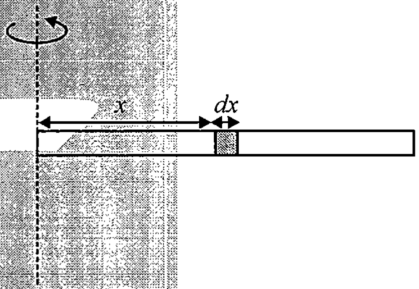
$$=\frac{ML^{2}}{12}\quad\therefore\quad\boxed{I=\frac{ML^{2}}{12}}$$

**Case 2:** About an axis through its end and perpendicular to its length

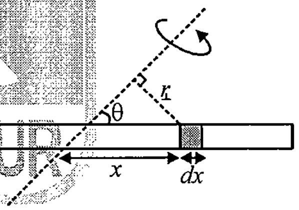
Here, $r_{\perp}=x$

$$\therefore\quad I=\int_{0}^{L}x^{2}\frac{M}{L}dx$$

$$=\frac{ML^{2}}{3}\quad\therefore\boxed{I=\frac{ML^{2}}{3}}$$



**Case 3:** About an axis passing through its centre and making an angle $\theta$ with its length.



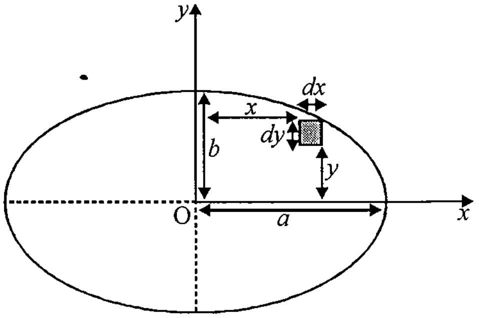
Here, $r_{\perp}=x\sin\theta$

$$\therefore\quad I=\int_{-L/2}^{L/2}x^{2}\sin^{2}\theta.\frac{M}{L}dx=\frac{M}{L}\sin^{2}\theta\int_{-L/2}^{L/2}x^{2}dx$$

$$=\frac{ML^{2}}{12}\sin^{2}\theta\quad\therefore\quad\boxed{I=\frac{ML^{2}}{12}\sin^{2}\theta}$$

**Moment of inertia of uniform elliptical thin plate:** Let M = mass of plate and $a$, $b$ be the semi-major and minor axes respectively.

If centre of ellipse is take as origin and major and minor axes as $x$ and $y$ axes respectively then equation of ellipse is given as $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$



Let us take an element at point ($x$, $y$). Let its area be $dx.dy$

therefore, $dm=\frac{M}{A}.dx.dy=\frac{M}{\pi ab}dx.dy$

where $A=\pi ab$ = area of ellipse.

**Case 1:** About the major axis

Here, $r_{\perp}=y$

$$\therefore \quad I=\int r_\perp^2 dm=\iint y^2.\frac{M}{\pi ab}dxdy \;=\frac{M}{\pi ab}\iint y^2 dx.dy=\frac{M}{\pi ab}\int dx\int y^2 dy$$

From equation of ellipse $y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

$$\therefore I=\frac{M}{\pi ab}\int_{-a}^{a}dx\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}y^2dy=\frac{2M}{3\pi ab}\int_{-a}^{a}dx\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^3=\frac{2Mb^2}{3\pi a}\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2}dx=\frac{4Mb^2}{3\pi a}\int_{0}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2}dx$$

The value of integral is $\frac{3\pi}{16}a$

Therefore, $I=\frac{4Mb^2}{3\pi a}.\frac{3\pi}{16}a=\frac{Mb^2}{4}$ or $\boxed{I=\frac{Mb^2}{4}}$ about major axis

**Case 2:** About the minor axis

Here $r_\perp=x \quad \therefore \; I=\int r_\perp^2 d_m=\iint x^2\frac{M}{\pi ab}dx\,dy$

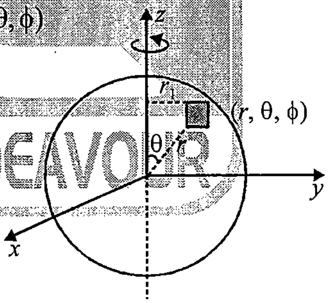
A similar calculation gives $\boxed{I=\frac{Ma^2}{4}}$ about minor axis

**Note:** It is easier to use plane polar, spherical polar and cylindrical coordinates for objects having circular symmetry, spherical symmetry and cylindrical symmetry. Therefore depending on the symmetry we will use convenient coordinate system.

**Moment of inertia of a uniform solid sphere about its diameter:** Let M = mass of the sphere and R = radius of sphere. Let us take an element at a point $(r,\theta,\phi)$

Volume of element $(dv)$ = $r^2\sin\theta d\theta d\phi dr$

Therefore, $dm=\frac{M}{V}dV \;=\frac{M}{\frac{4}{3}\pi R^3}r^2dr\sin\theta d\theta d\phi$



Here $r_\perp=r\sin\theta$ (as shown in figure)

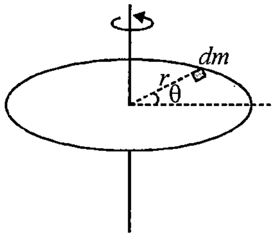
$$\therefore \quad I=\int r_\perp^2+dm=\iiint r^2\sin^2\theta.\frac{M}{\frac{4}{3}\pi R^3}r^2dr\sin\theta d\theta d\phi \;=\frac{3M}{4\pi R^3}\int_0^R r^4dr\int_0^\pi\sin^3\theta d\theta\int_0^{2\pi}d\phi$$

$$=\frac{3M}{4\pi R^3}.\frac{R^5}{5}.\frac{4}{3}.2\pi=\frac{2}{5}MR^2 \qquad \therefore \; \boxed{I=\frac{2}{5}MR^2}$$

**Moment of inertia of uniform circular disc:** Let M = mass of disc, R = radius of the disc, let us take an element at $(r,\theta)$. Area of element $(dA)=rdrd\theta$

Mass of element $(dm)$ = $\frac{M}{A}.dA=\frac{M}{\pi R^2}rdrd\theta$

here $r_\perp = r$

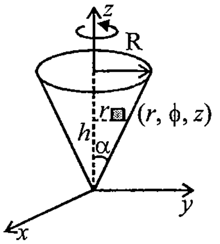


Therefore, $I = \int r_\perp{}^2 dm = \iint r^2 . \frac{M}{\pi R^2} r dr d\theta = \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\theta$

$= \frac{M}{\pi R^2} . \frac{R^4}{4} . 2\pi \qquad \therefore \boxed{I = \frac{MR^2}{2}}$

**Moment of inertia of uniform right circular cone about its axis:** Let M = mass, R = base radius, $h$ = height of the cone. Let us take an element at $(r, \phi, z)$ in cylindrical coordinate system.

Volume of element $(dV)$ = $rdrd\phi dz$



Mass of the element $(dm)$ = $\frac{M}{\frac{1}{3}\pi R^2 h} . rdrdzd\phi$

Here $r_\perp = r$

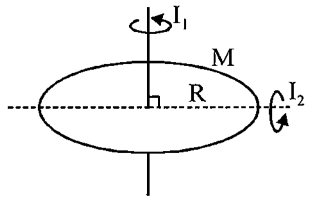
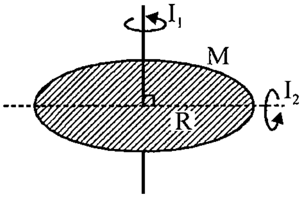
Therefore, $I = \int r_\perp^2 dm = \iiint r^2 . \frac{M}{\frac{1}{3}\pi R^2 h} rdrdz\phi = \frac{3M}{\pi R^2 h} \int dz \int r^3 dr \int_0^{2\pi} d\phi$

At the slanting surface of cylinder $\tan\alpha = \frac{r}{z}$ or $r = z\tan\alpha$

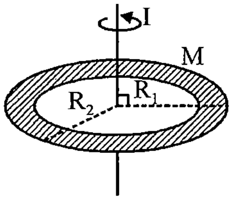
$\therefore \quad I = \frac{3M}{\pi R^2 h} \int_0^h dz \int_0^{z\tan\alpha} r^3 dr \int_0^{2\pi} d\phi = \frac{3M}{\pi R^2 h} . \int_0^h dz . \frac{z^4 \tan^4\alpha}{4} . 2\pi$

$= \frac{3M}{2R^2 h} \tan^4\alpha \int_0^h z^4 dz = \frac{3M}{2R^2 h} . \frac{R^4}{h^4} . \frac{h^5}{5} = \frac{3}{10} MR^2 \quad \therefore \boxed{I = \frac{3}{10} MR^2}$

**List of moment of inertia of some objects:**

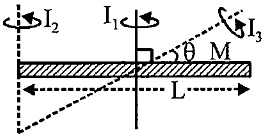
| | **Object** | **Moment of inertia** |
|---|---|---|
| Thin circular ring |  | $I_1 = MR^2, I_2 = \frac{MR^2}{2}$ |
| Thin circular disc |  | $I_1 = \frac{MR^2}{2}, I_2 = \frac{MR^2}{4}$ |

Annular ring



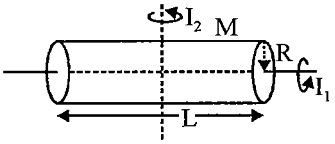
$$I=\frac{1}{2}M(R_1^2+R_2^2)$$

Thin rod



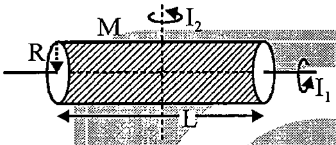
$$I_1\frac{ML^2}{12},I_2=\frac{ML^2}{3},I_3=\frac{ML^2}{12}\sin^2\theta$$

Thin hollow cylinder



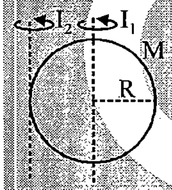
$$I_1=MR^2,I_2=M\left(\frac{R^2}{2}+\frac{L^2}{12}\right)$$

Solid cylinder



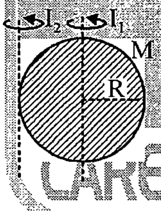
$$I_1=\frac{MR^2}{2},I_2=M\left(\frac{R^2}{4}+\frac{L^2}{12}\right)$$

Thin hollow sphere



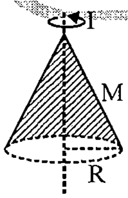
$$I_1=\frac{2}{3}MR^2,I_2=\frac{5}{3}MR^2$$

Solid sphere



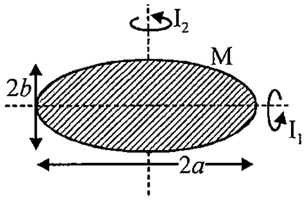
$$I_1=\frac{2}{5}MR^2,I_2=\frac{7}{5}MR^2$$

Right circular solid cone



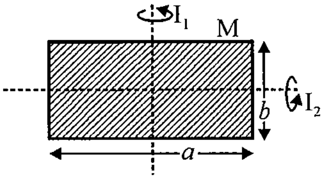
$$I=\frac{3}{10}MR^2$$

Elliptical plate



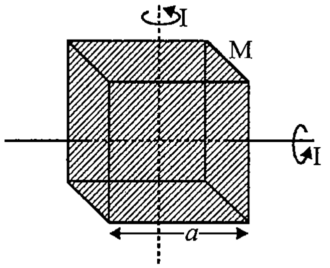
$$I_1=\frac{Mb^2}{4},I_2=\frac{Ma^2}{4}$$

Rectangular plate



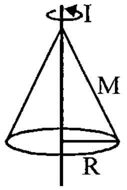
$$I_1=\frac{Ma^2}{12},I_2=\frac{Mb^2}{12}$$

Solid cube



$I = \frac{Ma^2}{6}$

Right circular hollow cone



$I = \frac{MR^2}{2}$

**Example :** A thin circular disc of mass 'M' and radius 'R' has non-uniform mass density which varies linearly with distance from centre. Calculate moment of inertia of the disc about its axis.

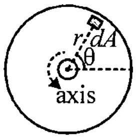
**Soln.** Let us taken an element at $(r, \theta)$

Area of element $(dA) = r dr d\theta$

Mass of element $(dm) = \sigma dA$

where $\sigma$ = surface mass density. According to question $\sigma \alpha r$.

Therefore, $\sigma = kr$, $k$ = constant



$\therefore \quad dm = \sigma dA = kr^2 dr d\theta$

$$\therefore \quad M = \iint kr^2 dr d\theta = k\int_0^R r^2 dr \int_0^{2\pi} d\theta = \frac{2\pi k R^3}{3} \quad \text{or} \quad k = \frac{3M}{2\pi R^3} \qquad \text{... (a)}$$

Moment of inertia about the axis of disc (axis passing through centre and perpendicular to its plane)

$$I = \int r_{\perp}^2 dm = \iint r^2\, kr^2 dr d\theta = k\int_0^R r^4 dr \int_0^{2\pi} d\theta = 2\pi k \frac{R^5}{5}$$

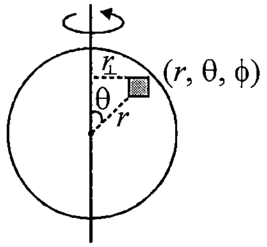
Using value of $k$ from (a) we get, $I = \frac{3}{5}MR^2$

**Example:** A solid sphere of mass M and radius R has non-uniform mass density which varies linearly with distance from centre. Calculate moment of inertia of the sphere about its diameter.

**Soln.** According to question, $\rho \alpha r$ or $\rho = kr$

Let us take an element at $(r, \theta, \phi)$

Mass of the element $dm = \rho dv = kr.r^2 dr \sin\theta d\theta d\phi$



$$\therefore M = \iiint kr^3 dr \sin\theta d\theta d\phi = k\int_0^R r^3 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \pi k R^4$$

$$\therefore \quad k = \frac{M}{\pi R^4} \qquad \text{... (a)}$$

Moment of inertia about its diameter

Here $r_\perp = r\sin\theta$

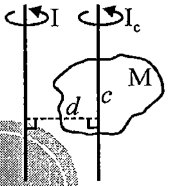
$$I = \int r_\perp^2 dm = \iiint r^2 \sin^2\theta . kr^3 dr \sin\theta . d\theta d\phi$$

$$= k\int_0^R r^5 dr \int_0^\pi \sin^3\theta d\theta . \int_0^{2\pi} d\phi \;=\; k\frac{R^6}{6}.\frac{4}{3}.2\pi \;=\; \frac{4\pi k R^6}{9}$$

using value of $k$ from (a) we get, $I = \dfrac{4MR^2}{9}$

**Parallel axes theorem:** It states that if $I_c$ be moment of inertia of an object about an axis passing through its centre of mass then moment of inertia about any other parallel axis which is at a separation '$d$' is given as

$$I = I_c + Md^2$$

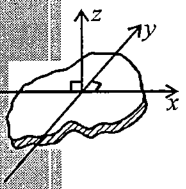


where M = mass of object

This theorem is valid for all types of objects ($1d$, $2d$, $3d$)

**Perpendicular axes theorem:** This theorem is applicable for $1d$ or $2d$ objects. It states that if $x$ and $y$ axes lie in the plane of object and $z$ axis is perpendicular to its plane then

$$I_{zz} = I_{xx} + I_{yy}$$



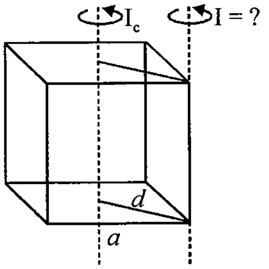
The three axes must intersect at common point, however there is no condition as such that the axes should pass through centre of mass. Therefore the axes may or may not pass through centre of mass.

**Example:** A uniform solid cube has mass M and side '$a$'. Calculate moment of inertia of the cube about an axis which coincides with one of its sides.

**Soln.** The axis given in question and a parallel axis through centre of mass are shown in the figure.

separation between two axes is $d = \dfrac{a}{\sqrt{2}}$

we know that moment of inertia of cube about an axis through its centre and perpendicular to its face is $I_c = \dfrac{Ma^2}{6}$



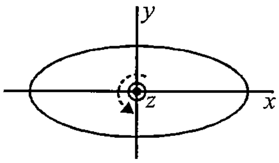
Therefore using parallel axes theorem we get

$$I = I_c + Md^2 = \frac{Ma^2}{6} + \frac{Ma^2}{2} = \frac{2}{3}Ma^2$$

**Example:** An elliptical plate has mass M and semi-major and minor axes '$a$' and '$b$' respectively. Calculate moment of inertia of the plate about an axis perpendicular to its plane and passing through its centre.

**Soln.** Let us take semi-major and minor axes as $x$ and $y$ axes and the axis given in the question as $z$ axis.

$\therefore \quad I_{xx} = \dfrac{Mb^2}{4}, I_{yy} = \dfrac{Ma^2}{4}$



From perpendicular axes theorem

$$I_{zz} = I_{xx} + I_{yy} = \frac{M}{4}(a^2 + b^2)$$

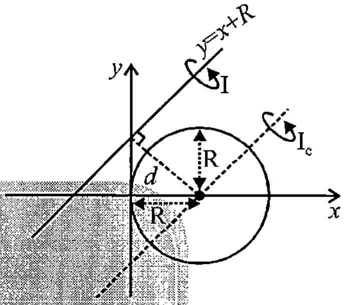
**Example:** A thin circular ring of mass M and radius R lies in *x*-*y* plane. Equation of the ring is $(x-R)^2 + y^2 = R^2$. Calculate moment of inertia of the ring about a straight line $y = x + R$.

**Soln.** The ring and the straight line are shown in the figure. To calculate moment of inertia about the given axis we draw a parallel line through centre of ring and apply parallel axes theorem.

$$\therefore I = I_c + Md^2$$

$$= \frac{MR^2}{2} + M(R\sqrt{2})^2$$

$$= MR^2 + 2MR^2 = 3MR^2$$



**Example:** Moment of inertia of a thin circular ring of unit mass lying in *x*-*y* plane is 2 unit and 3 unit respectively about the straight lines $y = x + 3$ and $y = -x + 2$. If centre of the ring lies at (1, 1) calculate moment of inertia of the ring about the axis perpendicular to its plane and passing through its centre.

**Soln.** Point of intersection of the two straight lines is $P\left(-\frac{1}{2}, \frac{5}{2}\right)$. Distance of point of intersection from centre of the circle is $d = \sqrt{\left(1+\frac{1}{2}\right)^2 + \left(1-\frac{5}{2}\right)^2} = \frac{3}{\sqrt{2}}$. Moment of inertia of the ring about an axis passing through point P and perpendicular to *x*-*y* plane is (use perpendicular axes theorem)

$$I = 2 + 3 = 5$$

The perpendicular axis through P is parallel to perpendicular axis through the centre of the ring. Therefore using parallel axis theorem we get

$$I = I_c + Md^2$$

$$\therefore \quad 5 = I_c + 1 \times \frac{9}{2} \quad \text{or} \quad I_c = 0.5$$

**Dynamics of rigid body:** A rigid body can have following types of dynamics

(i) Pure translation: If all the points of an object undergo equal displacement in equal interval of time then dynamics is called pure translation. Pure translational motion of a rigid body is equivalent to translation of centre of mass of the object. Therefore,

Kinetic energy = $\frac{1}{2}MV_c^2$, momentum = $MV_c$

Equation of motion $\vec{F} = M\vec{a}_c$

where $v_c$ and $a_c$ are velocity and acceleration of centre of mass.

(ii) Pure rotation: In this type of motion at least one point of rigid body remains stationary. Rotational motion of a rigid body cannot be described in terms of its outer of mass. Instead dynamics of each particle has to be considered. That is why concept of moment of inertia comes into play in rotational motion.

In pure rotation distance of each particle of rigid body remains constant from the point or axis about which rotation takes place.

In pure rotational motion we use following terms for a rigid body.

Kinetic energy $= \frac{1}{2}I\omega^2$, where I = moment of inertia about given axis

Angular momentum about the axis $L = I\omega$

Equation of motion $\tau = I\alpha$ (about the axis)

where $\alpha = \frac{d\omega}{dt}$, $\alpha$ = angular acceleration, $\omega$ = angular velocity

and $\omega = \frac{d\theta}{dt}$, $\theta$ = angular displacement

If $\alpha$ = constant then $\omega = \omega_0 + \alpha t, \omega^2 = \omega_0^2 + 2\alpha\theta, \theta = \omega_0 t + \frac{1}{2}\alpha t^2$

where $\omega_0$ = initial angular velocity, $\omega$ = final angular velocity

(iii) General motion: In this case object undergoes translation as well rotation. There is no point which remains stationary for all the time. And different points of the object undergo different displacement. According to Chasle's theorem a general motion is equivalent to translation of centre of mass plus pure rotation about centre of mass i.e.

General motion = Pure translation of centre of mass + Pure rotation about centre of mass

Therefore in general motion we use following terms for a rigid body.

Kinetic energy $= \frac{1}{2}MV_c^2 + \frac{1}{2}I_c\omega^2$

Equation of motion $F = Ma_c$, $\tau = I_c\alpha_c$

Angular momentum $\vec{L} = I_c\vec{\omega} + M\vec{r} \times \vec{v}_c$. If the rigid body is round in shape and it is moving on a surface then about any point on the path of the object angular momentum is written as $L = I_c\omega + Mv_cR$ where R is radius of object.

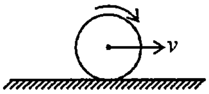
(iv) Pure rolling: It is a special kind of general motion of round objects in which following relations are satisfied.

$$\left.\begin{aligned} v_c &= \omega R \\ a_c &= \alpha R \end{aligned}\right\} \text{condition for rolling}$$

In most cases friction is necessary for rolling motion. However there is no loss of energy against friction in rolling motion. Therefore total mechanical energy remains conserved in rolling motion all though friction is present.

**Example:** A hollow sphere of mass M and radius R rolls on a horizontal surface such that centre of the sphere moves with speed $v$. Calculate kinetic energy and angular momentum of the sphere.

**Soln.** Given, $v_c = v$ and sphere is rolling



Therefore we use condition of rolling $(v_c = \omega R)$

To find its angular velocity

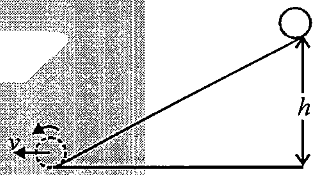
$$\therefore \omega = \frac{v_c}{R} = \frac{v}{R}$$

$$\text{Kinetic energy} = \frac{1}{2}Mv_c^2 + \frac{1}{2}I_c\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}.\frac{2}{3}MR^2.\frac{v^2}{R^2} = \frac{5}{6}Mv^2$$

$$\text{Angular momentum} = I_c\omega + Mv_cR = \frac{2}{3}MR^2.\frac{v}{R} + MvR = \frac{5}{3}MvR$$

**Example:** A hollow sphere rolls down an inclined plane from a height '$h$' what is speed of the sphere when it reaches the bottom of inclined plane.

**Soln.** There must be friction on the plane. However total mechanical energy is conserved in rolling motion. Therefore, loss in potential energy = gain in kinetic energy



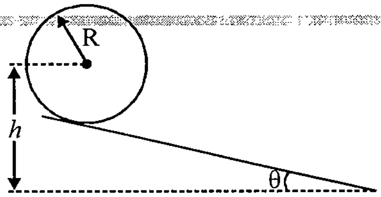
$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

for rolling $\omega = v/r$, for hollow sphere $I = \frac{2}{3}mr^2$

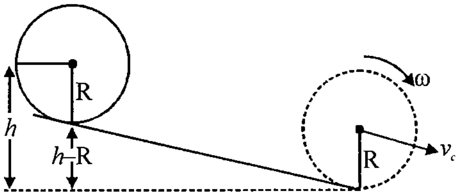
$$\therefore\ mgh = \frac{1}{2}mv^2 + \frac{1}{2}.\frac{2}{3}mr^2.\frac{v^2}{r^2} \Rightarrow mgh = \frac{5}{6}mv^2$$

$$\therefore v = \sqrt{\frac{6gh}{5}}$$

**Example:** A ring of mass $m$ and radius $R$ rolls (without slipping) down an inclined plane starting from rest. If the centre of the ring is initially at a height $h$, the angular velocity when the ring reaches the base is



**Soln.** Height descended by centre of mass of the ring is $h - R$.
From conservation of energy we get,
loss in potential energy = gain in K.E.



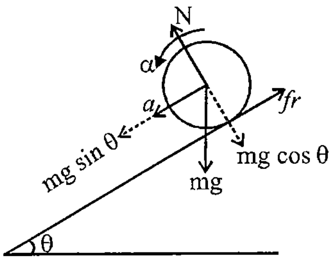
$$mg(h-R) = \frac{1}{2}mv_c^2 + \frac{1}{2}I\omega^2$$

for rolling $v_c = \omega R, I = mR^2$

$$\therefore\ mg(h-R) = \frac{1}{2}m\omega^2R^2 + \frac{1}{2}mR^2\omega^2 \Rightarrow mg(h-R) = m\omega^2R^2, \quad \therefore \omega = \sqrt{\frac{g(h-R)}{R^2}}$$

**Acceleration of rolling object on inclined plane:** Let us consider an object of mass M, radius R and moment of inertia I about the axis about which it rotates while rolling.

Let $a$ and $\alpha$ be linear and angular acceleration of the object. Therefore for rolling



$a = \alpha R$ ... (i)

Equation of translational motion $F = ma$

or $mg\sin\theta - f_r = ma$ ... (ii)

Equation of rotational motion $\tau = I\alpha$

or $f_r R = I.\alpha$

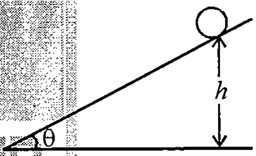
or $f_r = \dfrac{Ia}{R^2}$ ... (iii)

(ii) + (iii) gives $mg\sin\theta = a\left(m + \dfrac{I}{R^2}\right)$ or $\boxed{a = \dfrac{g\sin\theta}{1 + \dfrac{I}{mR^2}}}$

from (iii) we get $\boxed{f_r = \dfrac{I}{R^2}.\dfrac{g\sin\theta}{1 + \dfrac{I}{mR^2}}}$

friction force in rolling motion is static in nature. Therefore if $\mu$ be coefficient of static friction then

$f_r \le \mu N$ or $\dfrac{I}{R^2}.\dfrac{g\sin\theta}{1 + \dfrac{I}{mR^2}} \le \mu.mg\cos\theta$ or $\boxed{\mu \ge \dfrac{\tan\theta}{1 + \dfrac{mR^2}{I}}}$



**Example:** A solid sphere rolls down on inclined plane from a height '$h$' as shown in figure. Calculate time taken by sphere to reach the bottom. Also calculate the minimum value of coefficient of friction necessary for rolling.

**Soln.** Moment of inertia of solid sphere $I = \dfrac{2}{5}mr^2$

acceleration of sphere $a = \dfrac{g\sin\theta}{1 + \dfrac{I}{mr^2}} = \dfrac{g\sin\theta}{1 + \dfrac{\frac{2}{5}mr^2}{mr^2}} = \dfrac{5g\sin\theta}{7}$

distance covered along the inclined plane $= \dfrac{h}{\sin\theta}$

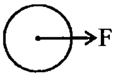
$\therefore$ If $t$ be time taken then, $S = \dfrac{1}{2}at^2$

or $\dfrac{h}{\sin\theta} = \dfrac{1}{2}.\dfrac{5g\sin\theta}{7}.t^2$ $\therefore t = \sqrt{\dfrac{14h}{5g\sin^2\theta}}$

minimum value of coefficient of friction necessary to support rolling is

$$\mu = \frac{\tan\theta}{1+\frac{mr^2}{I}} = \frac{\tan\theta}{1+\frac{mr^2}{\frac{2}{5}mr^2}} = \frac{2}{7}\tan\theta$$

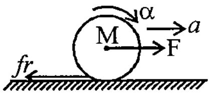
**Example :** A constant force F is applied horizontally at the centre of a circular disc of radius R and mass M as shown in the figure. If the disc rolls on the surface without sliding then. Calculate its acceleration.



**Soln.** There must be friction between the disc and surface to support rolling. Let *fr* be force of friction. Therefore, translational equation of motion. $\vec{F}_{net} = m\vec{a}$

$$f - f_r = Ma \qquad \text{... (a)}$$

Rotational equation of motion $\tau = I.\alpha$



$$f_r R = \frac{2}{3}MR^2.\alpha$$

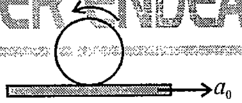
for pure rolling $a = \alpha R$

$$\therefore \; f_r R = \frac{2}{3}MR^2.\frac{a}{R}$$

$$\text{or} \quad f_r = \frac{2}{3}Ma \qquad \text{... (b)}$$

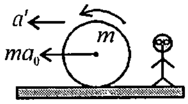
(a) + (b) gives, $F = \frac{5}{3}Ma \quad \therefore \; a = \frac{3F}{5M}$

**Example:** A hollow sphere is kept on a plank which moves horizontally as shown in the figure. The sphere rolls on the plank. Calculate acceleration of the sphere with respect to ground.



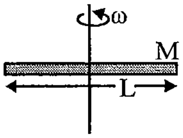
**Soln.** Let us first find acceleration of sphere with respect to the plank. For this we observe motion of sphere from the plank's frame. Therefore a pseudo force has to be applied on the sphere. Now this is exactly the same problem as the previous one with applied force being equal to $ma_0$ i.e. $F = ma_0$. Therefore acceleration of sphere with respect to the plank will be

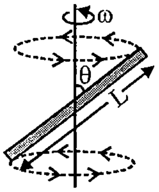
$$a' = \frac{3F}{5m} = \frac{3ma_0}{5m} = \frac{3}{5}a_0$$



Therefore acceleration of sphere with respect to the ground is $a = a_0 - \frac{3}{5}a_0 = \frac{2}{5}a_0$

**Rotational about a fixed point:** The formulae for angular momentum and kinetic energy $L = I\omega$ and kinetic energy $= \frac{1}{2}I\omega^2$ can be used if rotation is taking place about a fixed axis. In fact in some cases $L = I\omega$ represents angular momentum about the axis only, it is not equal to total angular momentum. One such situation is shown in figure below. In the figure we have shown two different rotational motion of a thin rod about a fixed axis





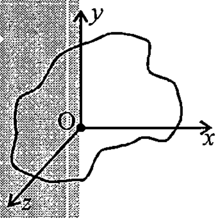
Angular momentum about the axis =

$$I\omega = \frac{ML^2}{12}\omega = \text{Total angular momentum}$$

Angular momentum about the axis

$$= I\omega = \frac{ML^2}{12}\sin^2\theta\omega \neq \text{total angular momentum}$$

However, if rotation is taking place about a fixed point (not about fixed axis) then above mentioned formulae for angular momentum and kinetic energy can not be used. Let O be a fixed point about which a rigid body is rotating. Let us take a set of three perpendicular axes through O as shown in the figure. The axes taken are supposed to be fixed with the rigid body. Therefore w.r.t. to lab the orientation of axes keep changing. These are called body axes and are taken as instantaneous axes of rotation. If $\omega_x, \omega_y$ and $\omega_z$ be components of angular velocity and $L_x, L_y$ and $L_z$ components of angular momentum at some instant of time then value of angular momentum is calculated from following matrix relation.



$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

or $\boxed{\bar{L} = \overline{II}.\bar{\omega}}$

and rotational kinetic energy is written as

$$K = \frac{1}{2}(\omega_x \omega_y \omega_z)\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \text{ or } \boxed{K = \frac{1}{2}\vec{\omega}.\overline{II}.\vec{\omega}} \quad \boxed{K = \frac{1}{2}\vec{\omega}.\vec{L}}$$

$$\text{where } \overline{II} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \text{inertia tensor}$$

The element of inertia tensor are written as

| | Elements | Discrete case | Continuous case |
|---|---|---|---|
| Moment of inertia | $I_{xx} =$ | $\sum_i m_i(y_i^2 + z_i^2)$ | $\int (y^2 + z^2)dm$ |
| | $I_{yy} =$ | $\sum_i m_i(x_i^2 + z_i^2)$ | $\int (x^2 + z^2)dm$ |
| | $I_{zz} =$ | $\sum_i m_i(x_i^2 + y_i^2)$ | $\int (x^2 + y^2)dm$ |

| | Elements | Discrete case | Continuous case |
|---|---|---|---|
| Product of inertia | $I_{xy}=I_{yx}=$ | $-\sum_i m_i x_i y_i$ | $-\int xydm$ |
| | $I_{xz}=I_{zx}=$ | $-\sum_i m_i x_i z_i$ | $-\int xzdm$ |
| | $I_{yz}=I_{zy}=$ | $-\sum_i m_i y_i z_i$ | $-\int yzdm$ |

**Principal axes:** A set of three axes so chosen that all the product of inertia becomes zero, are called principal axes. Therefore inertia tensor becomes diagonal if axes chosen happen to be principal axes.

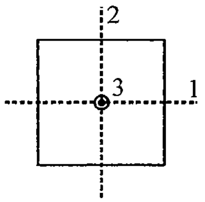
Thus, if $x, y, z$ are principal axes then $I_{ij}=0$ if $i \neq j$

Therefore, $\overleftrightarrow{I}=\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

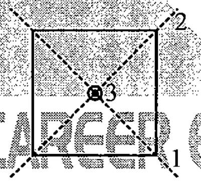
In this case $I_{xx}, I_{yy}, I_{zz}$ are called principal moment of inertia. Thus principal moment of inertia are moment of inertia about principal axes. Principal axes are usually denoted by numbers 1, 2 and 3. Accordingly principal moment of inertia are denoted as $I_1$, $I_2$ and $I_3$.

- Symmetric axes of an object are always principal axes.
- Principal axes may or may not be symmetric axes.
- Through each point of an object a set of three principal axes can be defined. Therefore an object can have infinite number of principal axes.

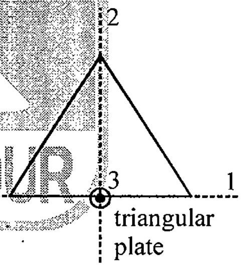
In the figure below principal axes of some objects are shown



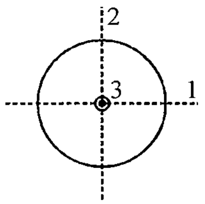
Square plate



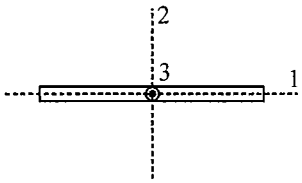
Square plate



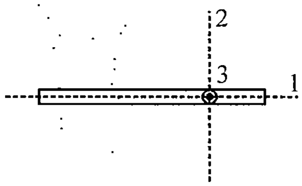
triangular plate



Sphere



Thin rod

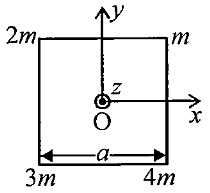


Thin rod

**Finding principal moment of inertia and principal axes:** Eigen values of inertia tensor are equal to principal moment of inertia and eigen vectors of inertia tensor give orientation of principal axes.

Therefore, to find principal axes through a point of an asymmetric object we can choose any three convenient axes and write inertia tensor and to get principal axes we need to calculate eigen vectors.

**Example:** Four point masses $m, 2m, 3m$ and $4m$ are placed at the corners of a square of side '$a$' as shown in the figure. Write inertia tensor for the axes shown in figure and find the principal axes passing through point O.



**Soln.** Coordinates of masses are, $m\left(\frac{a}{2},\frac{a}{2},0\right)$, $2m\left(\frac{-a}{2},\frac{a}{2},0\right)$, $3m\left(\frac{-a}{2},\frac{-a}{2},0\right)$, $4m\left(\frac{a}{2},\frac{-a}{2},0\right)$

therefore, $I_{xx}=\sum_i m_i(y_i^2+z_i^2)=m\left(\frac{a^2}{4}\right)+2m\left(\frac{a^2}{4}\right)+3m\left(\frac{a^2}{4}\right)+4m\left(\frac{a^2}{4}\right)=\frac{5}{2}ma^2$

$$I_{yy}=\sum m_i(x_i^2+z_i^2)=m\left(\frac{a^2}{4}\right)+2m\left(\frac{a^2}{4}\right)+3m\left(\frac{a^2}{4}\right)+4m\left(\frac{a^2}{4}\right)=\frac{5}{2}ma^2$$

$$I_{zz}=\sum m_i(x_i^2+y_i^2)=5ma^2$$

$$I_{xy}=-\sum m_i x_i y_i=-m\left(\frac{a}{2}\right).\left(\frac{a}{2}\right)-2m\left(\frac{-a}{2}\right).\left(\frac{a}{2}\right)-3m\left(\frac{-a}{2}\right)\left(\frac{-a}{2}\right)-4m\left(\frac{a}{2}\right)\left(\frac{-a}{2}\right)$$

$$=\frac{ma^2}{4}(-1+2-3+4)=\frac{ma^2}{2}=I_{yx}$$

$$I_{xz}=I_{zx}=-\sum m_i x_i z_i=0\ (\because z_i=0),\quad I_{yz}=I_{zy}=-\sum m_i y_i z_i=0\ (\because z_i=0)$$

therefore, $I\!I=ma^2\begin{pmatrix}5/2 & 1/2 & 0\\ 1/2 & 5/2 & 0\\ 0 & 0 & 5\end{pmatrix}$

Principal moment of inertia: Let $\lambda$ be eigen value of inertia tensor.

$$\therefore\ \begin{vmatrix}\frac{5ma^2}{2}-\lambda & \frac{ma^2}{2} & 0\\ \frac{ma^2}{2} & \frac{5ma^2}{2}-\lambda & 0\\ 0 & 0 & \frac{5ma^2}{2}-\lambda\end{vmatrix}=0 \qquad \therefore\left(\frac{5ma^2}{2}-\lambda\right)\left[\left(\frac{5ma^2}{2}-\lambda\right)^2-\left(\frac{ma^2}{2}\right)^2\right]=0$$

$$\therefore\ \frac{5ma^2}{2}-\lambda=0 \quad \text{or} \quad \left(\frac{5ma^2}{2}-\lambda\right)^2-\left(\frac{ma^2}{2}\right)^2=0$$

$$\therefore\quad \lambda = \frac{5ma^2}{2} \qquad \text{or} \qquad \frac{5ma^2}{2} - \lambda = \pm\frac{ma^2}{2} \Rightarrow \lambda = 2ma^2, 3ma^2$$

$\therefore\ \lambda = 2ma^2, 3ma^2, 5ma^2$ (Principal moment of inertia). Let $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$ be eigen vector corresponding to eigen value $2ma^2$.

Therefore,

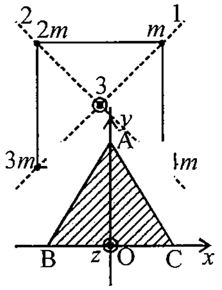
$$\begin{pmatrix} \frac{5ma^2}{2} & \frac{ma^2}{2} & 0 \\ \frac{ma^2}{2} & \frac{5ma^2}{2} & 0 \\ 0 & 0 & 5ma^2 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = 2ma^2\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}, \qquad \begin{pmatrix} \frac{5ma^2}{2}\alpha + \frac{ma^2}{2}\beta \\ \frac{ma^2}{2}\alpha + \frac{5ma^2}{2}\beta \\ 5ma^2\gamma \end{pmatrix} = \begin{pmatrix} 2ma^2\alpha \\ 2ma^2\beta \\ 2ma^2\gamma \end{pmatrix}$$

$\therefore\quad 5\alpha + \beta = 4\alpha \quad$ or $\quad \alpha = -\beta$

If $\alpha = 1$ then $\beta = -1$

Therefore, normalized eigen vector corresponding to $\lambda = 2ma^2$ is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$.

Similarly we find that normalized eigen vector corresponding to $\lambda = 3ma^2$ is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}$.



Thus principal axes for given mass distribution are as shown in figure:

**Example:** A uniform equilateral triangular plate of mass 'M' and side '$a$' lies in $x$-$y$ plane as shown in figure. Write inertia tensor.

**Soln.** $y$-axis is a symmetric axis therefore it must be a principal axis. And $x$-being perpendicular to $y$, is also a principal axis.

Therefore, $x-y-z$ axes given in figure are principal axes. Therefore all product of inertia are zero.

And, $I_{xx} = \int (y^2 + z^2)dm = \int y^2 dm \quad (\because z = 0)$

$$= \iint y^2 . \frac{M}{\frac{\sqrt{3}}{4}a^2} . dxdy = \frac{4M}{\sqrt{3}a^2}\int dx \int y^2 dy$$

$$=\frac{8M}{\sqrt{3}\,a^2}\int_0^{a/2}dx\int_0^{\frac{\sqrt{3}}{2}(a-2x)}y^2dy \qquad \{\because \text{ equation of AC is } y=-\sqrt{3}x+\frac{\sqrt{3}}{2}a=\frac{\sqrt{3}}{2}(a-2x)\}$$

$$=\frac{M}{a^2}\int_0^{a/2}dx(a-2x)^3 \;=\frac{M}{a^2}\left[\frac{-\frac{1}{2}(a-2x)^4}{4}\right]_0^{a/2}=\frac{Ma^2}{8}$$

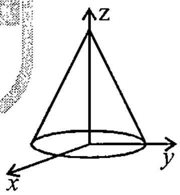
$$I_{yy}=\int(x^2+z^2)dm=\int x^2dm=\iint x^2.\frac{M}{\frac{\sqrt{3}}{4}a^2}dx.dy=\frac{4M}{\sqrt{3}a^2}\iint x^2dxdy=\frac{8M}{\sqrt{3}a^2}\int_0^{a/2}x^2dx\int_0^{\frac{\sqrt{3}}{2}(a-2x)}dy$$

$$=\frac{4M}{a^2}\int_0^{a/2}x^2(a-2x)dx=\frac{4M}{a^2}\left[\frac{a.\left(\frac{a}{2}\right)^3}{3}-\frac{2.\left(\frac{a}{2}\right)^4}{4}\right]=\frac{4M}{a^2}\left[\frac{a^4}{24}-\frac{a^4}{32}\right]=\frac{Ma^2}{24}$$

$$I_{zz}=\int(x^2+y^2)dm=I_{xx}+I_{yy}=\frac{Ma^2}{8}+\frac{Ma^2}{24}=\frac{Ma^2}{6}$$

Therefore, $II=Ma^2\begin{pmatrix}1/8 & 0 & 0\\ 0 & 1/24 & 0\\ 0 & 0 & 1/6\end{pmatrix}$

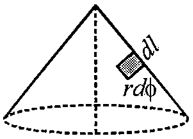
**Example :** A hollow right circular cone has mass M, radius R, height H. Find inertia tensor of the cone about the axes shown in the figure.

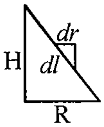


**Soln.** $x, y, z$ are symmetric axes. Therefore, these are principal axes.
So the inertia tensor is diagonal i.e. all product of inertia must be zero.
let us use cylindrical coordinate system

$\therefore\;\; x=r\cos\phi, y=r\sin\phi, z=z$

Area element on the surface of cone $= rd\phi dl=rd\phi.\frac{\sqrt{R^2+H^2}}{R}dr$



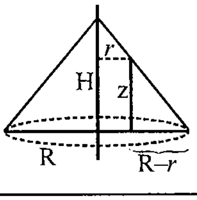


Area of slanting surface of cone $=\pi R\sqrt{R^2+H^2}$

$$\therefore\;\; dm=\frac{M}{\pi R\sqrt{H^2+R^2}}.rd\phi\frac{\sqrt{R^2+H^2}}{R}dr=\frac{M}{\pi R^2}.rdrd\phi$$

$$I_{xx}=\int(y^2+z^2)dm=\iint(r^2\sin^2\phi+z^2).\frac{M}{\pi R^2}rdrd\phi$$

Consider a point on surface of cone as shown in figure



using similarity of triangles we get

$$\frac{z}{R-r}=\frac{H}{R} \quad \therefore\ z=H\left(1-\frac{r}{R}\right)$$

$$\therefore I_{xx}=\iint\left[r^2\sin^2\phi+H^2\left(1-\frac{r}{R}\right)^2\right]\frac{M}{\pi R^2}.rdrd\phi$$

$$=\frac{M}{\pi R^2}\left[\int_0^R r^3dr\int_0^{2\pi}\sin^2\phi d\phi+H^2\int_0^R\left(1-\frac{r}{R}\right)^2 rdr\int_0^{2\pi}d\phi\right]=\frac{M}{\pi R^2}\left[\frac{R^4}{4}.\pi+H^2\frac{R^2}{12}.2\pi\right]=\frac{M}{4}\left[R^2+\frac{2H^2}{3}\right]$$

$x$ and $y$ axes are equivalent. Therefore,

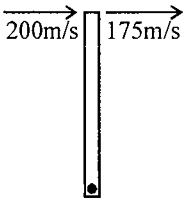
$$I_{yy}=\frac{M}{4}\left(R^2+\frac{2H^2}{3}\right)$$

$$I_{zz}=\int(x^2+y^2)dm=\iint(r^2\cos^2\phi+r^2\sin^2\phi).\frac{M}{\pi R^2}rdrd\phi$$

$$=\frac{M}{\pi R^2}\int_0^R r^3dr\int_0^{2\pi}d\phi=\frac{MR^2}{2} \qquad \therefore II=\begin{pmatrix}\frac{M}{4}\left(R^2+\frac{2H^2}{3}\right) & 0 & 0\\ 0 & \frac{M}{4}\left(R^2+\frac{2H^2}{3}\right) & 0\\ 0 & 0 & \frac{MR^2}{2}\end{pmatrix}$$

## SOLVED PROBLEMS

1. A bullet of mass 0.1kg travelling at a speed of 200m/s hits one end of a stick of length 1.0m and mass 0.5kg lying on a horizontal table pivoted at the other end as shown in the figure. After hitting the stick the bullet continues to move along its original path but with a reduced speed of 175 m/s. Assume there is no friction.



(a) Find angular speed of stick after it is hit

(b) Find the magnitude of the impulse on the pivot because of the hit.

**Soln.** (a) If ω be the angular velocity of the stick just after collision then from conservation of angular momentum about the pivot we get,

$$0.1\times200\times1=0.1\times175\times1+\frac{0.5\times1^2}{3}\omega$$

$\therefore\ \omega=15$ rad/sec

(b) Let J be impulse on the rod due pivot

Therefore, using total impulse on rod = change in momentum of the rod we get $0.1\times(200-175)+J=0.5\times v_c$ (i)

$$2.5+J=0.5v_c \quad \text{(ii)}$$

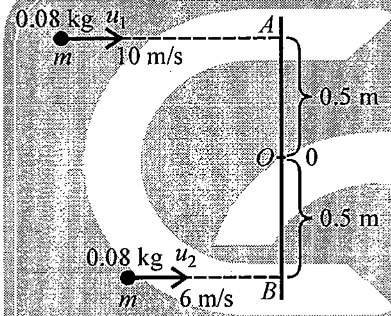
where $v_c$ is velocity of centre of mass of rod after being hit.

$$\therefore \quad v_c=\omega\times\frac{1}{2}=\frac{15}{2}$$

$$\therefore \quad \text{from (i)} \quad 2.5+J=\frac{7.5}{2}\Rightarrow J=1.25$$

Therefore, impulse on pivot is $-1.25$ kgm/sec

2. A thin uniform bar lies on a frictionless horizontal surface and is free to move in any way on the surface. Its mass is 0.16 kg and length is $\sqrt{3}$ m. Two particles, each of mass 0.08 kg are moving on the same surface and towards the bar in a direction perpendicular to the bar, one with a velocity of 10 m/sec and other with 6 m/sec as shown in figure. The first particle strikes the bar at a point *A* and the other at point *B*. Points *A* and *B* are at a distance of 0.5 m from the center of the bar. The particles strike the bar at the same instant of time and stick to the bar no collision. Calculate the loss of kinetic energy of the system in the above collision process.



**Soln.** As the particles after the collision stick to the rod, the center of mass of the system coincides with the mid-point of the rod, because $OA=OB=0.5$ m as the masses of the particles being equal. During collision pair forces develop and the rod is free to rotate as well as translate, both total momentum and total angular momentum will be conserved.

Applying the conservation of linear momentum

$$mu_1+mu_2=(M+2m)v_{CM}, \quad \text{...(i)}$$

where *M* is the mass of the bar.

Substituting the given values, we have

$$(0.08\times10+0.08\times6)=(0.16+2\times0.08)v_{CM}$$

Solving we get, $v_{CM}=4$ m/sec.

Taking angular momentum about center of bar, we get

$$mv_1\times y-mv_2\times y=I\omega. \quad \text{... (ii)}$$

$$\text{Here } I=\left[\frac{Ml^2}{12}+2my^2\right]=\left[\frac{0.16\times\left(\sqrt{3}\right)^2}{12}+2\times0.08\times(0.5)^2\right]$$

$$\therefore \quad I=0.08\text{ kg m}^2$$

Substituting the given values in equation (ii), we get

$$0.08\times10\times0.5-0.08\times6\times0.5=0.08\,\omega.$$

Solving we get, $\omega = 2\ \text{rad/sec}^2$.

Loss of kinetic energy, $\Delta E = \frac{1}{2} m u_1^2 + \frac{1}{2} m u_2^2 - \frac{1}{2}(M + 2m) v_{CM}^2 + \frac{1}{2} I \omega^2$

$$\therefore \quad \Delta E = \frac{1}{2} \times 0.08 \times (10)^2 + \frac{1}{2} \times 0.08 \times (6)^2 - \frac{1}{2}(0.16 + 2 \times 0.08) \times (4)^2 + \frac{1}{2}(0.08) \times (2)^2$$

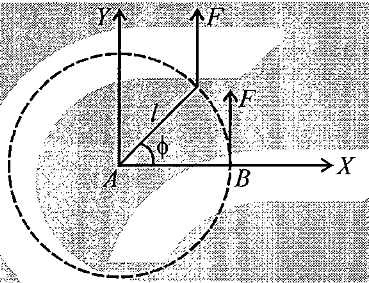
$= \mathbf{2.72\ joule.}$

3. A thin horizontal uniform rod $AB$ of mass $m$ and length $l$ can rotate freely about a vertical axis passing through its end $A$. At a certain moment the end $B$ starts experiencing a constant force $F$ which is always perpendicular to the original position of the stationary rod and directed in the horizontal plane. Find the angular velocity of the rod as a function of its rotation angle $\phi$ counted relative to the initial position.

**Soln.** The situation of the rod at an angle $\phi$ is shown in figure.

Here, $r = il \cos\phi + jl \sin\phi$ and $F = jF$ (Force is always perpendicular to rod)

$\vec{\tau} = r \times F = (il \cos\phi + jl \sin\phi) \times (jF) = lF \cos\phi\, k$ or $|\vec{\tau}| = lF \cos\phi$



We know that $\tau = I\alpha$

Here, $I = \frac{1}{3} ml^2$ (for rod) and $\alpha = \omega\left(\frac{d\omega}{d\phi}\right)$

Therefore, $lF \cos\phi = \frac{1}{3} ml^2 \cdot \omega\left(\frac{d\omega}{d\phi}\right)$ or $lF \cos\phi\, d\phi = \frac{1}{3} ml^2 \cdot \omega\, d\omega$

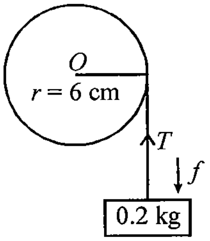
Integrating within proper limits, we have

$$lF \int_0^{\phi} \cos\phi\, d\phi = \frac{1}{3} ml^2 \int_0^{\omega} \omega\, d\omega$$

$$lF \sin\phi = \frac{1}{3} ml^2 \left(\frac{\omega^2}{2}\right)$$

$$\therefore \quad \omega = \sqrt{\left[\frac{6F \sin\phi}{ml}\right]}.$$

4. A wheel of radius 6 cm is mounted so as to rotate about a horizontal axis through its center. A string of negligible mass wrapped round its circumference carries a mass of 0.2 kg attached to its free end. When let fall, the mass descends through one meter in 5 seconds. Calculate the angular acceleration of the wheel, its moment of inertia and tension in the cord.



**Soln.** Let $f$ be the acceleration of the falling mass, then using the formula $h = ut + \frac{1}{2}at^2$, we have $1 = 0 \cdot t + \frac{1}{2}f(5)^2$

Therefore, $f = \frac{2}{25} = 0.08$ m/sec$^2$.

Now angular acceleration, $\alpha = \frac{\text{linear acceleration}}{r} = \frac{0.08}{6 \times 10^{-2}} = \mathbf{1.33\ rad./sec^2}$.

After 5 seconds,
The velocity of wheel $v = ft = 0.08 \times 5$ m/sec. $= 0.4$ m/sec.

Angular velocity of wheel $\omega = \frac{v}{r} = \frac{0.4}{6 \times 10^{-2}} = 6.66$ rad/sec.

Potential energy lost by falling mass $= mgh$.

Increase in translational kinetic energy $= \frac{1}{2}mv^2$.

Increase in rotational energy $= \frac{1}{2}I\omega^2$. (where $I$ is the moment of inertia)

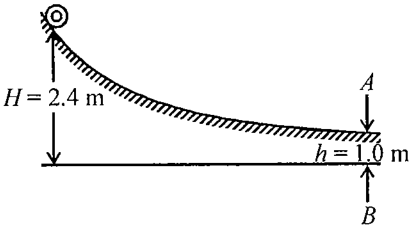
Therefore, $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$ or $\frac{1}{2} \times 0.2 \times (0.4)^2 + \frac{1}{2}I(6.66)^2 = 0.2 \times 2.8 \times 1$.

Solving we get $I = \mathbf{8.747 \times 10^{-2}\ kg\text{-}m^2/sec.}$

Let $T$ be the tension in the string. Hence, $mf = mg - T$ or $T = m(g - f)$

Therefore, $T = 0.2(9.8 - 0.08) = \mathbf{1.9445}$.

5. A small sphere rolls down without slipping from the top of a track in a vertical plane. The track has an elevated section and a horizontal part. The horizontal part is 1.0 meter above the ground level and the top of the track is 2.4 meters above the ground. Find the distance on the ground with respect to point $B$ [which is vertically below the end of the track as shown in figure, where the sphere lands. During its flight as a projectile, does the sphere continue to rotate about its center of mass ? Explain.



**Soln.** The loss of potential energy when the cylinder comes from the top of the track to point $A$

$$= mg(H-h) = mg(2.4-1.0) = 1.4\,mg.$$

The gain in its energy $= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$, where $v$ and $\omega$ are the linear and angular velocities of cylinder.

Here $I = \frac{2}{5}mr^2$ and $\omega = \left(\frac{v}{r}\right)$.

Therefore, gain in its kinetic energy $= \frac{1}{2}mv^2 + \frac{1}{2}\times\left(\frac{2}{5}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{7}{10}mv^2$.

Now, loss of potential energy = gain in kinetic energy $1.4\,mg = \frac{7}{10}mv^2$.

Therefore, $v^2 = \frac{10\times 1.4\times 9.8}{7} = 19.6$ m/sec or $v = \sqrt{(19.6)}$ m/sec.

This is the horizontal velocity at point $A$. The vertical component is zero.

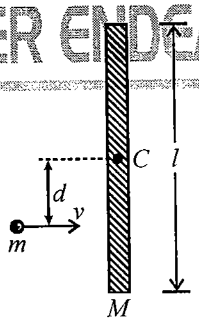
Now, time taken to fall a distance 1 m is $t = \sqrt{\left(\frac{2h}{g}\right)} = \sqrt{\left(\frac{2\times 1}{9.8}\right)}$.

Distance of the position of landing the ball on ground from $B$.

$$s = vt = \sqrt{(19.6)}\times\sqrt{\left(\frac{2\times 1}{9.8}\right)} = 2\,\text{m}$$

Yes, The sphere rotates about its center of mass so that the law of conservation of angular momentum holds good during its flight.

6. A meter stick lies on a frictionless horizontal table. It has a mass $M$ and is free to move in any way on the table. A small body of mass $m$ moving with speed $v$ as shown in figure collides elastically with the stick. What must be the value of $m$ if it is to remain at rest after the collision ?



**Soln.** The collision is elastic. Therefore, linear momentum, angular momentum about a point (say center of mass $C$) and mechanical energy should be conserved. Let $v'$ be the linear velocity acquired by the stick after collision and $\omega$ be the angular velocity about $C$.

By conservation of linear momentum $mv = Mv'$ ... (i)

By conservation of angular momentum

Angular momentum of body at $C$ = Angular momentum of stick about $C$

$$md^2\cdot\left(\frac{v}{d}\right) = \frac{Ml^2}{12}\cdot\omega \qquad \text{... (ii)}$$

[where angular momentum of body about $C = I\omega = md^2 \cdot \left(\dfrac{v}{d}\right)$ and $\dfrac{Ml^2}{12}$ is the moment of inertia of stick about $C$].

By the conservation of mechanical energy

$$\frac{1}{2}mv^2 = \frac{1}{2}Mv'^2 + \frac{1}{2}\left(\frac{Ml^2}{12}\right)\omega^2 \qquad \text{... (iii)}$$

From equation (i), $v' = \left(\dfrac{mv}{M}\right)$

From equation (ii), $\omega = (12\,mvd/Ml^2)$

Substituting these values in equation (iii), we get

$$mv^2 = M\left(\frac{mv}{M}\right)^2 + \left(\frac{Ml^2}{12}\right)\left(\frac{12\,mvd}{Ml^2}\right)^2$$

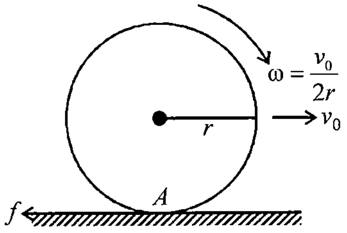
$$1 = \frac{m}{M} + \left(\frac{Ml^2}{12}\right)\left(\frac{12d}{Ml^2}\right)^2 m$$

or $$1 = \frac{m}{M} + md^2\left(\frac{12}{Ml^2}\right)$$

or $$1 = \frac{m}{M}\left(1 + \frac{12d^2}{l^2}\right) = \frac{m}{M}\left(\frac{l^2 + 12d^2}{l^2}\right)$$

$$m = \frac{Ml^2}{(l^2 + 12d^2)}.$$

7. A sphere of mass $M$ and radius $r$ shown in figure slips on a rough horizontal plane. At some instant, it has translational velocity $v_0$ and rotational velocity about the center $v_0/2r$. Find the translational velocity after the sphere starts pure rolling.



**Soln.** The velocity of the center $= v_0$

Angular velocity about center $= \dfrac{v_0}{2r}$.

Here $v_0 > \omega_0 r$. Hence the sphere slips forward. The friction by the plane on the sphere will act backward. The frictional force $= \mu Mg$. The deceleration of the sphere $a_{CM} = \dfrac{f}{M}$.

$$\therefore \quad v(t) = v_0 - \frac{f}{M}t \qquad \text{...(i)}$$

Torque of frictional force $\tau = f \times r$ (clockwise)

Angular acceleration about center $\alpha = f\dfrac{r}{\left(\dfrac{2}{5}\right)Mr^2} = \dfrac{5f}{2Mr}$

Angular velocity at time $t$

i.e., $\omega(t) = \omega_0 + \dfrac{5f}{2Mr}t$ or $\omega(t) = \dfrac{v_0}{2r} + \dfrac{5f}{2Mr}$ or $\left[\dfrac{v_0}{r}\right] = \dfrac{v_0}{2r} + \dfrac{5ft}{2Mr}$

$$v(t) = \frac{v_0}{2} + \frac{5ft}{2M} \qquad \text{...(ii)}$$

From equations (i) and (ii), eliminating $t$, we get

$$\frac{5}{2}v(t) + v(t) = \frac{5}{2}v_0 + \frac{v_0}{2};\ v(t) = \frac{6}{7}v_0$$

Thus, the sphere rolls with translational velocity $\dfrac{6v_0}{7}$ in the forward direction.

8. A solid cylinder (a) rolls (b) slides from rest down an inclined plane. Neglect friction and compare the velocities in both cases when cylinder reaches the bottom of the incline.

**Soln.** (a) We know that the acceleration $a$ of the body rolling down an inclined plane is given by $a = \dfrac{g\sin\theta}{(1 + K^2/R^2)}$.

For a solid cylinder $K^2 = \dfrac{R^2}{2}$, hence $a = \dfrac{2g\sin\theta}{3}$.

Let $v_1$ be the velocity acquired by the cylinder on reaching the bottom of the incline of length $l$, then

$$v_1^2 = 0 + 2al = \frac{4g\sin\theta \cdot l}{3} \qquad \text{...(i)}$$

(b) When the cylinder slides down the plane, the acceleration $a = g\sin\theta$.

If $v_2$ be the velocity acquired by the cylinder on reaching the bottom, then

$$v_2^2 = 0 + 2al = 2g\sin\theta \cdot l \qquad \text{...(ii)}$$

Now, $\dfrac{v_2^2}{v_1^2} = \dfrac{2g \cdot \sin\theta \cdot l}{\left(4g \cdot \sin\theta \cdot \dfrac{l}{3}\right)} = \dfrac{3}{2}$ or $\dfrac{v_2}{v_1} = \sqrt{\left(\dfrac{3}{2}\right)} = \mathbf{1.225.}$

9. A sphere, a disc and a ring of the same mass and radius are allowed to roll down an inclined plane simultaneously from the same height without slipping. Prove that the sphere reaches down first, the disc next and the ring, the last.

**Soln.** We know that acceleration $a$ of the body rolling down an inclined plane is given by

$$a = \frac{g\sin\theta}{\left(1 + \dfrac{K^2}{R^2}\right)} \qquad \text{...(i)}$$

We also know that, Moment of inertia of solid sphere $I_1 = M\,K_1^2 = \frac{2}{5} M\,R^2$.

$$\therefore \qquad K_1^2 = \frac{2}{5} R^2 \qquad \text{... (ii)}$$

Moment of inertia of disc $I_2 = M\,K_2^2 = \frac{M\,R^2}{2}$.

$$\therefore \qquad K_2^2 = \frac{R^2}{2} \qquad \text{... (iii)}$$

Moment of inertia of ring $I_3 = M\,K_3^2 = M\,R^2$.

$$\therefore \qquad K_3^2 = R^2 \qquad \text{... (iv)}$$

Substituting these values in equation (i), we get

Acceleration $a_1$ for solid sphere $= \frac{g\sin\theta}{(1+2/5)} = \frac{5}{7} g\sin\theta$.

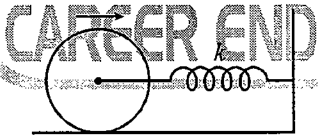
Acceleration $a_2$ for disc $= \frac{g\sin\theta}{(1+2/5)} = \frac{2}{3} g\sin\theta$.

Acceleration $a_3$ for ring $= \frac{g\sin\theta}{(1+1)} = \frac{g}{2}\sin\theta$.

$$\therefore \qquad a_1 > a_2 > a_3$$

Thus, solid sphere reaches down first, the disc next and the ring, the last.

10. A solid cylinder is attached to a horizontal massless spring, so that it can roll without slipping along a horizontal surface. The force constant $k$ of the spring is 3.0 nt/meter. If the system is released from rest at a position in which the spring is stretched 0.25 meter, find the translational kinetic energy and the rotational kinetic energy of the cylinder as it passes through the equilibrium position.



**Soln.** (a) Let the linear speed of the center of mass at the equilibrium position be $v$ and angular speed $\omega$. Since the cylinder rolls without slipping, $v = r\,\omega$, where $r$ is the radius of cylinder.

Translational kinetic energy $= \frac{1}{2} m\,v^2$.

Rotational kinetic energy $= \frac{1}{2} I\,\omega^2 = \frac{1}{2}\cdot\left(\frac{1}{2} m\,r^2\right)\left(\frac{v^2}{r^2}\right) = \frac{1}{4} m\,v^2$.

Thus, rotational K.E. $= \frac{1}{2}$ translational K.E.

Total energy $= \frac{1}{2} m\,v^2 + \frac{1}{4} m\,v^2 = \frac{3}{4} m\,v^2$.

This should be equal to the total potential energy stored in the spring when it is stretched 0.25 meter and the cylinder released from rest.

Therefore, $\frac{3}{4}mv^2 = \frac{1}{2}k(0.25)^2 = \frac{1}{2}\times 3.0\times(0.25)^2$ or $mv^2 = \frac{1}{2}\times 3.0\times(0.25)^2\times\left(\frac{4}{3}\right) = 0.125$ joules.

So, Translational kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}(0.125) =$ **0.0625 joules**

Rotational kinetic energy $= \frac{1}{4}mv^2 = \frac{1}{4}(0.125) =$ **0.03125 joules.**

11. A carpet of mass $M$ made of inextensible material is rolled along its length in the form of a cylinder of radius $R$ and is kept on a rough floor. The carpet starts unrolling without sliding on the floor when a negligibly small push is given to it. Calculate the horizontal velocity of the axis of the cylindrical part of the carpet when its radius reduces to $R/2$.

**Soln.** Let $l$ be the length of rolled carpet. The mass $M$ of the rolled carpet is given by

$$M = \text{Volume} \times \text{density} = \pi R^2 l \times \rho$$

Let $M_1$ be the mass of the rolled carpet when its radius is reduced to $(R/2)$. Thus,

$$M_1 = \frac{M}{\pi R^2 l\rho}\times\pi\left(\frac{R}{2}\right)^2 l\rho = \frac{M}{4}$$

Potential energy of cylinder before rolling $= MgR$.

For the mass $\left(\frac{M}{4}\right)$, the center of gravity is at $\frac{R}{2}$.

Hence change in potential energy $= MgR - \left(\frac{M}{4}\right)g\left(\frac{R}{2}\right) = \frac{7}{8}MgR$

Now, change in P.E. = linear kinetic energy + rotational K.E.

$$\therefore \quad \frac{7}{8}MgR = \frac{1}{2}M_1v^2 + \frac{1}{2}I\omega^2$$

where $v$ is the horizontal velocity of the cylindrical part of the carpet. Now,

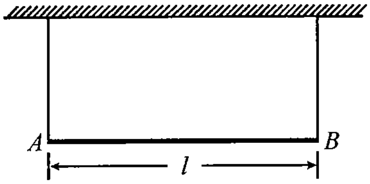
$$\frac{7}{8}MgR = \frac{1}{2}\left(\frac{M}{4}\right)v^2 + \frac{1}{2}\left[\frac{1}{2}M_1\left(\frac{R}{2}\right)^2\right]\left[\frac{v}{R/2}\right]^2$$

$$\text{or} \quad \frac{7}{8}MgR = \frac{1}{2}\left(\frac{M}{4}\right)v^2 + \frac{1}{64}MR^2\times\frac{4v^2}{R^2}$$

$$\text{or} \quad \frac{7}{8}gR = \frac{v^2}{8} + \frac{v^2}{16} = \frac{3v^2}{16}$$

$$\text{or} \quad v = \left[\left(\frac{14}{3}\right)gR\right]^{1/2} = \mathbf{6.623\sqrt{R}\ m/s.}$$

12. A uniform rod of length $l$ and mass $M$ is suspended on two vertical inextensible strings as shown in figure. Calculate tension $T$ in left string at the instant, when right string snaps.

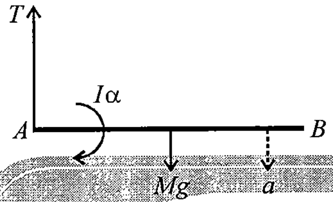


**Soln.** When the right string snaps, the following two forces are acting on the rod :

(i) weight $Mg$ and

(ii) tension $T$ in left string.

These two forces produce a moment and the rod rotates about its left end as shown in figure.



For vertical forces, we have $Mg - T = Ma$ ...(i)

If $\alpha$ be the angular acceleration of the rod, then the downward acceleration of its center $a$ is given by

$$a = \left(\frac{l}{2}\right)\alpha$$

Substituting the value of $a$ in equation (i), we get $Mg - T = M\left(\frac{l}{2}\right)\alpha$ ...(ii)

Taking moments about $A$

$$Mg = \left(\frac{l}{2}\right) = I\alpha = \left(\frac{Ml^2}{3}\right)\alpha$$

$$\alpha = \frac{3g}{2l} \quad \text{...(iii)}$$

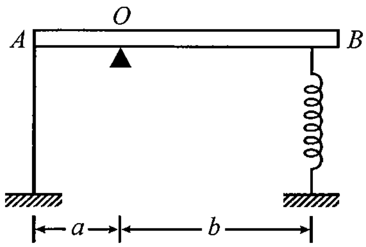
Substituting the value of $\alpha$ from equation (iii) in equation (ii), we get

$$Mg - T = M\left(\frac{l}{2}\right)\frac{3g}{2l}$$

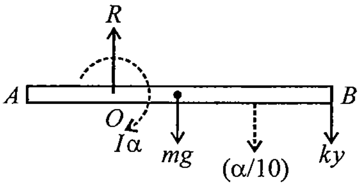
$$Mg - \frac{3Mg}{4} = T$$

$$\therefore \quad T = \frac{Mg}{4}.$$

13. A uniform rod $AB$ of mass 2 kg and length $l = 100$ cm is placed on a sharp support $O$ such that $AO = a = 40$ cm and $OB = b = 60$ cm. A spring of force constant $k = 600$ N/m is attached to end $B$ as shown in figure. To keep the rod horizontal, its end $A$ is tied with a thread such that the spring is elongated by $y = 1$ cm. Calculate reaction of support on the rod when the thread is burnt.



**Soln.** The different forces on rod are shown in figure.



Due to these forces ($Mg$ and $ky$) the rod is subjected to a clockwise angular acceleration. As a result, the center of mass of the rod accelerates vertically downwards. If $\alpha$ be the angular acceleration of the rod, then downward acceleration of center of mass will be

$$\left(\frac{l}{2}-a\right)\alpha = 0.1\,\alpha = \left(\frac{\alpha}{10}\right) \quad \text{...(i)}$$

Taking moments about $O$, we have $mg\times(0.10)+ky(0.60) = I\alpha$ ...(ii)

where $I$ = moment of inertia of the rod about O

$$= \frac{ml^2}{12}+m\left(\frac{l}{2}-a\right)^2 = \frac{2}{12}+2\times(0.1)^2 = 0.167+0.02 \approx 0.19 \text{ kg-m}^2$$

From equation (ii)

$$2\times10\times0.10+600\times(0.01)\,(0.60) = 0.19\,\alpha$$

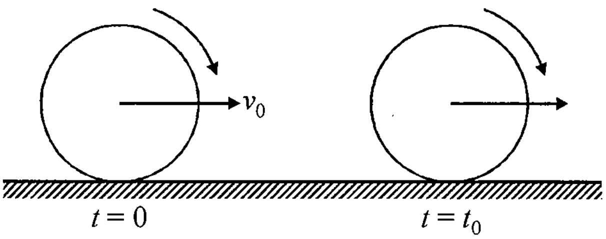
Solving we get $\alpha = 30$ rad/s$^2$

For rod, $mg+ky-N = m\left(\dfrac{\alpha}{10}\right)$

$$2\times10+600(0.01)-R = 2\times3$$

$R =$ **20 newton.**

14. A uniform disc of mass $m$ and radius $R$ is projected horizontally with velocity $v_0$ on a rough horizontal floor so that it starts with a purely sliding motion at $t=0$. After $t_0$ seconds it acquires a purely rolling motion as shown in figure.



(i) Calculate the velocity of the center of mass of the disc at $t_0$

(ii) Assuming the coefficient of friction to be $\mu$, calculate $t_0$. Also calculate the workdone by the frictional force as a function of time and the total workdone by it over a time $t$ much longer than $t_0$

**Soln.** (i) Maximum frictional force $= \mu mg$, therefore acceleration $= \mu g$.

For linear motion of center of mass $v = v_0 - \mu g t_0$ ... (i)

Here $v$ = velocity of center of mass at $t = t_0$

For purely rolling motion,

$$v = \omega R \quad \text{or} \quad v = \left[\frac{\mu mg R}{(mR^2/2}t_0\right]R \quad \text{or} \quad v = 2\mu g t_0 \qquad \text{... (ii)}$$

Substituting the value of $\mu g t_0$ from equation (ii) in equation (i), we get

$$v = v_0 - \frac{v}{2} \quad \text{or} \quad v = \left(\frac{2v_0}{3}\right) \qquad \text{... (iii)}$$

(ii) Substituting the value of $v$ from equation (iii) in equation (ii), we get

$$\frac{2v_0}{3} = 2\mu g t_0 \quad \text{or} \quad t_0 = \frac{v_0}{3\mu g} \qquad \text{... (iv)}$$

Let $W$ be the workdone by frictional force in time $t$, then

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m(v_0 - \mu g t)^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{2\mu g t}{R}\right) + W$$

or $$W = \frac{1}{2}mv_0^2 - \frac{1}{2}m\left[v_0^2 - 2v_0\mu g t + \mu^2 g^2 t^2\right] - m\mu^2 g^2 t^2$$

or $$W = \frac{1}{2}m\mu g t\left[2v_0 - 3\mu g t\right]$$

The workdone by frictional force for $t = t_0$

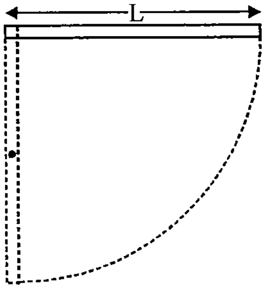
$$= \frac{1}{2}m\mu g\left(\frac{v_0}{3\mu g}\right)\left[-v_0 + 2v_0\right] = \frac{1}{6}mv_0 \times v_0 = \frac{mv_0^2}{6}.$$

After $t_0$, when the motion becomes pure rotational, there will be no workdone by frictional force.

So, for $t \gg t_0$, the total workdone by frictional force $= \frac{mv_0^2}{6}$.

15. A uniform beam held horizontally at two ends is set into motion by releasing it at one end, due to which it rotates about the other end. When rod becomes vertical the second end is also released. Find the trajectory of centre of rod and angular velocity of rod in subsequent motion.

**Soln.** Let L be length and M be mass of the rod. Let $\omega$ be the angular speed of the rod when the second end is released.



$\therefore$ From conservation of energy we get

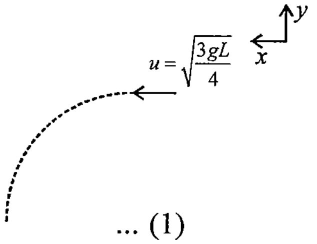
$$\frac{1}{2}\cdot\left(\frac{ML^2}{3}\right)\omega^2 = Mg\cdot\frac{L}{2}$$

$$\therefore \quad \omega = \sqrt{\frac{3g}{L}}$$

Therefore, velocity of centre of rod at this moment

$$u = \omega \cdot \frac{L}{2} = \sqrt{\frac{3gL}{4}}$$

Now centre of the rod moves like a particle falling under gravity.
Therefore its path will be parabolic.
To find equation of path let us assume that at an instant '$t$' the co-ordinate of the centre is $(x, y)$.



$$\therefore \quad x = ut \qquad \text{... (1)}$$

and $y = \frac{1}{2} g t^2$

To get equation of path we eliminate $t$ from the above two equations

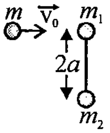
$$\therefore \quad y = \frac{1}{2} g \left(\frac{x}{u}\right)^2 \text{ or } y = \frac{g}{2} \cdot \frac{x^2}{\frac{3gL}{4}}$$

$y = \frac{2}{3}\frac{x^2}{L}$ (with origin L/2 below the second support)

Angular velocity of rod in subsequent motion will be same as the angular velocity of rod at the time of the release of second support.

16. A particle of mass $m$ strikes a dumbell and comes to rest as shown in the figure. Calculate the distance moved by the centre of mass of dumbell in the time dumbell rotates through $\pi/2$.

**Soln.** Distance of centre of mass of dumbel from $m_1$ is



$$X = \frac{m_2 a}{m_1 + m_2}$$

M.I. of dumbell about centre of mass

$$I_C = \frac{m_1 \cdot m_2}{m_1 + m_2} \cdot a^2$$

From conservation of momentum we get $mv_0 = (m_1 + m_2)v$

where $v$ is velocity of centre of mass of dumbel after collision

$$\therefore \quad v = \frac{m_1 v_0}{m_1 + m_2}$$

Now using conservation of angular momentum we get.

$$mv_0 X = I_C \omega$$

where $\omega$ is angular velocity of dumbel about centre of mass after collision.

$$\therefore \quad \omega = \frac{m}{m_1} \cdot \frac{v_0}{a}$$

Let '$t$' be time for rotation by $\pi/2$

$$\therefore \quad \pi/2 = \omega t$$

$$\therefore \quad t = \frac{2}{\pi}\cdot\frac{m_1}{m_1}\cdot\frac{a}{v_0}$$

$\therefore$ Distance covered in time $t = vt = \frac{2}{\pi}\cdot\frac{m_1 a}{m_1+m_2}$

17. A uniform rod of mass M and length L is rotating with angular velocity $\omega$ about an axis passing through its centre and $\perp$ to its length. If suddenly one end of the rod is fixed, calculate new angular velocity.

**Soln.** Since, torque about the end is zero, angular momentum is conserved.

Initial angular momentum of the rod.

$$L_i = \frac{ML^2}{12}\omega$$

This is also the initial angular momentum of the rod about its one end.

Moment of inertia of rod about its end is $\frac{ML^2}{3}$

Let $\omega'$ be the final angular velocity

$\therefore$ Final angular momentum $L_f = \frac{ML^2}{3}\cdot\omega'$

Since $L_i = L_f$

$$\therefore \quad \frac{ML^2}{12}\omega = \frac{ML^2}{3}\omega' \qquad \therefore \quad \omega' = \frac{\omega}{4}$$

18. A solid body rotates with deceleration about a stationary axis with an angular deceleration $\beta \propto \sqrt{\omega}$, where $\omega$ is its angular velocity. Find the mean angular velocity of the body averaged over the whole time of rotation if at the initial moment of time its angular velocity was equal to $\omega_0$.

**Soln.** $-\beta = k\sqrt{\omega}$ or $-\frac{d\omega}{dt} = k\sqrt{\omega}$ $\int_{\omega_0}^{0}\frac{d\omega}{\sqrt{\omega}} = -\int_0^t k\,dt$ $\therefore$ $t = \frac{2\sqrt{\omega_0}}{k}$

Average angular velocity $\overline{\omega} = \frac{\int \omega(t)\,dt}{\int dt}$

From $2\sqrt{\omega} = -kt,\ \omega = \frac{k^2t^2}{4}$

Therefore, $\overline{\omega} = \frac{\frac{k^2}{4}\int_0^{2\sqrt{\omega_0}/k} t^2\,dt}{\int_0^{2\sqrt{\omega_0}/k} dt} = \frac{k^2}{12}\left[\frac{t^3}{t}\right]_0^{2\sqrt{\omega_0}/k} = \frac{k^2}{12}\left[\frac{4\omega_0}{k^2}\right] = \frac{\omega_0}{3}$.

19. A solid body rotates about a stationary axis according to the law $\phi = kt - bt^3$, where $k = 6.0$ rad/s and $b = 2.0$ rad/s². Find

(a) The mean values of the angular velocity and angular acceleration averaged over the time interval between $t = 0$ and the complete stop;

(b) The angular acceleration at the moment when the body stops.

**Ans.** (a) $\bar{\omega} = 4$ rad/sec, $\bar{\beta} = 6$ rad/sec$^2$

(b) $\beta = 12$ rad/sec

**Soln.** $\phi = kt - bt^3$

$$\omega = \frac{d\phi}{dt} = k - 3bt^2; \quad \frac{d\omega}{dt} = 0 - 6bt$$

For stopping $\omega = 0$, then $t = \sqrt{\left(\frac{k}{3b}\right)}$

$$\text{Now, } \bar{\omega} = \frac{\int_0^1 \omega(t)\,dt}{\int_0^t dt} = \frac{\int_0^{\sqrt{k/3b}} (k - 3bt^2)\,dt}{\int_0^{\sqrt{k/3b}} dt} = \frac{\left[kt - (3bt^3/3)\right]_0^{\sqrt{k/3b}}}{\sqrt{k/3b}}$$

Solving we get $\bar{\omega} = 4$ rad/sec.

$$\text{Now, } \bar{\beta} = \frac{\int_0^t \beta(t)\,dt}{\int_0^t dt} = \frac{\int_0^{\sqrt{k/3b}} 6bt\,dt}{\int_0^{\sqrt{k/3b}} dt}$$

Solving we get $\bar{\beta} = 6$ rad/sec.

Further $\beta = -6bt = -6b\sqrt{k/3b}$

$$|\beta| = -6b\sqrt{(k/3b)} = 2\sqrt{(3kb)} = 12 \text{ rad/sec}^2.$$

20. A solid body rotates about a stationary axis so that its angular velocity depends on the rotation angle $\phi$ as $\omega = \omega_0 - k\phi$, where $\omega_0$ and k are positive constants. At the moment $t = 0$, the angle $\phi = 0$. Find the time dependent of

(a) rotation angle

(b) angular velocity

**Soln.** (a) Now rotation angle at time $t$:

$$\phi(t) = \int_{t=0}^{t} \omega\,dt = \int_0^1 \omega_0 e^{-kt} = -\frac{\omega_0}{k} e^{-kt}\Big|_0^t = \frac{\omega_0}{k}(1 - e^{-kt})\Big].$$

(b) Angular acceleration

$$\alpha = \frac{d\omega}{dt} = -\frac{d\phi}{dt} = -k\omega \text{ or } \frac{d\omega}{\omega} = -k\,dt$$

$$\int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = -k\int_0^t dt = -kt \text{ or } \omega = \omega_0 e^{-kt}.$$

21. Inertia tensor of a system is $\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$. Calculate moment of inertia about an axis which is oriented in the direction $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

**Soln.** Given $\overleftrightarrow{I} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}$ and $\hat{n} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\therefore \quad I = \hat{n}.\overleftrightarrow{I}.\hat{n}$$

$$= \frac{1}{3}(1 \quad 1 \quad 1)\begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3}(1 \quad 1 \quad 1)\begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} = \frac{1}{18}(1 \quad 1 \quad 1)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{18}(1+1+1) = \frac{1}{6}$$

22. Four equal point masses $m$ are placed the corners of a rectangle of sides '$a$' and '$b$'. The system is rotated about a diagonal with uniform angular velocity $\omega$. Calculate rotational kinetic energy, angular momentum about the axis of rotation and total angular momentum of system.

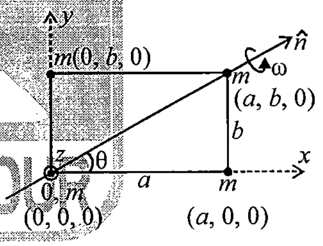
**Soln.** Let us choose $x$-$y$-$z$ axes as shown in figure.

$$I_{xx} = \sum m_i(y_i^2 + z_i^2) = \sum m_i y_i^2 \quad (\because z_i = 0)$$

$$= m0^2 + m0^2 + mb^2 + mb^2 = 2mb^2$$

$$I_{yy} = \sum_i m_i(x_i^2 + z_i^2) = \sum m_i x_i^2 = m0^2 + ma^2 + m0^2 + ma^2 = 2ma^2$$

$$I_{zz} = \sum_i m_i(x_i^2 + y_i^2) = 2m(a^2 + b^2)$$



$$I_{xy} = -\sum m_i x_i y_i = -mab = I_{yx} \qquad \therefore \; \overleftrightarrow{I} = m\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}$$

Direction of diagonal,

$$\hat{n} = \cos\theta\hat{i} + \sin\theta\hat{j} = \frac{1}{\sqrt{a^2+b^2}}(a\hat{i} + b\hat{j}) \equiv \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

moment of inertia about the axis of rotation

$$I = \hat{n}.\overleftrightarrow{I}.\hat{n} = \frac{m}{(a^2+b^2)}(a \quad b \quad 0)\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$= \frac{m}{a^2+b^2}\begin{pmatrix} a & b & 0 \end{pmatrix}\begin{pmatrix} ab^2 \\ a^2b \\ 0 \end{pmatrix} = \frac{2ma^2b^2}{(a^2+b^2)}$$

Therefore, rotational kinetic energy $= \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2ma^2b^2}{(a^2+b^2)}\omega^2 = \frac{ma^2b^2\omega^2}{a^2+b^2}$

Angular momentum about the axis of rotation $= I\omega = \frac{2ma^2b^2\omega}{(a^2+b^2)}$

To calculate total angular momentum of system we use $\vec{L} = \vec{\vec{I}}.\vec{\omega}$

or $\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$, Here $\omega_x = \omega\cos\theta = \frac{\omega a}{\sqrt{a^2+b^2}}$, $\omega_y = \omega\sin\theta = \frac{\omega b}{\sqrt{a^2+b^2}}$, $\omega_z = 0$

$$\therefore \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \frac{m\omega}{\sqrt{a^2+b^2}}\begin{pmatrix} 2b^2 & -ab & 0 \\ -ab & 2a^2 & 0 \\ 0 & 0 & 2(a^2+b^2) \end{pmatrix}\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = \frac{m\omega}{\sqrt{a^2+b^2}}\begin{pmatrix} ab^2 \\ a^2b \\ 0 \end{pmatrix}$$

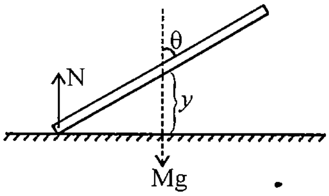
$$\therefore L_x = \frac{m\omega ab^2}{\sqrt{a^2+b^2}},\ L_y = \frac{m\omega a^2b}{\sqrt{a^2+b^2}}, L_z = 0$$

$$\therefore L = \sqrt{L_x^2+L_y^2+L_z^2} = \frac{m\omega ab}{\sqrt{a^2+b^2}}\sqrt{a^2+b^2}$$

$$\therefore L = m\omega ab$$

23. A rod of mass M and length L held vertically upward with its one end touching a smooth horizontal surface. The rod is released from this position. Show that when it is about to strike the surface, normal reaction is $Mg/4$.

**Soln.** Let at some instant of time the rod makes angle $\theta$ with the vertical as shown in the figure.



$$\therefore\ y = \frac{L}{2}\cos\theta$$

$$\dot{y} = -\frac{L}{2}\sin\theta\dot{\theta}$$

$$\ddot{y} = -\frac{L}{2}\sin\theta\ddot{\theta} - \frac{L}{2}\cos\theta\dot{\theta}^2 \qquad \text{... (i)}$$

downward acceleration of the rod $= -\ddot{y}$

$\therefore$ Equation of motion $Mg - N = -M\ddot{y}$ ... (ii)

For angular motion $\tau = \frac{ML^2}{12}\ddot{\theta}$ (about centre of rod)

$$\frac{NL}{2}\sin\theta=\frac{ML^2}{12}\ddot{\theta} \qquad \text{... (iii)}$$

using equations (i), (ii) and (iii) we get

$$Mg-N=+M\left[\frac{L}{2}\sin\theta\left(\frac{6N}{ML}\sin\theta\right)+\frac{L}{2}\cos\theta\dot{\theta}^2\right]$$

when rod is about to strike the horizontal surface

$\theta=90^\circ,\ \cos 90=0,\ \sin 90=1$

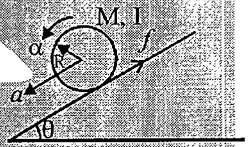
$$\therefore\ Mg-N=M\left[\frac{3N}{M}\times 1+0\right]$$

$$Mg=4N$$

$$\therefore\ N=\frac{Mg}{4}$$

24. Show that least coefficient of friction of a solid sphere to roll down an inclined plane is $\frac{2}{7}\tan\theta$ and that for a hollow sphere is $\frac{2}{5}\tan\theta$.

**Soln.** Consider an inclined plane of inclination $\theta$ and an object of mass M, radius R, and moment of inertia I rolling down on it.



Equation of motion;

$$I\alpha=f\cdot R$$

for rolling $\alpha=\dfrac{a}{R}$ $\qquad \therefore\ f=\dfrac{Ia}{R^2}$ ... (i)

and $\quad Mg\sin\theta-f=Ma$

using (i) we get $\ a=\dfrac{g\sin\theta}{1+\frac{I}{MR^2}}$

$$\therefore\ f=\frac{I}{R^2}\cdot\frac{g\sin\theta}{1+\frac{I}{MR^2}} \qquad N=Mg\cos\theta$$

since $f\le\mu N$

$$\therefore\ \frac{I}{R^2}\cdot\frac{g\sin\theta}{1+\frac{I}{(MR^2)}}\le\mu.Mg\cos\theta \qquad \therefore\ \mu\ge\frac{g\tan\theta}{1+\frac{MR^2}{I}}$$

For solid sphere

$$I=\frac{2}{5}MR^2 \qquad \therefore\ \mu\ge\frac{2}{7}g\tan\theta$$

For hollow sphere

$$I=\frac{2}{3}MR^2 \qquad \therefore\ \mu\ge\frac{2}{5}g\tan\theta$$

25. Calculate the radius of Gyration of a uniform solid cylinder of length L and radius R about an axis passing through its centre and perpendicular to its axis.

**Soln.** Moment of inertia of a solid cylinder about the given axis is

$$I = \frac{M}{4}\left(R^2 + \frac{L^2}{3}\right)$$

Radius of Gyration ($k$) is defined as

$$k = \sqrt{\frac{I}{M}} \qquad \therefore\ k = \frac{1}{2}\sqrt{R^2 + \tfrac{L^2}{3}}$$

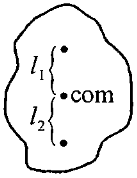
26. A physical pendulum has same period of oscillations T about two pivots which are L distance apart. Show that $T = 2\pi\sqrt{\frac{L}{g}}$.

**Soln.** Let $l_1$ and $l_2$ be the distances of the two pivots from the centre of mass of the physical pendulum $l_1 + l_2 = L$

$$\therefore\ T = 2\pi\sqrt{\frac{(I_c + Ml_1^2)}{Mgl_1}} \quad \text{and} \quad T = 2\pi\sqrt{\frac{(I_c + Ml_2^2)}{Mgl_2}}$$

where $I_C$ is moment of inertia about centre of mass and M is mass of physical pendulum.

$\therefore$ The above two expressions can be re-written as



$$\frac{T^2Mg}{4\pi^2}l_1 = I_C + Ml_1^2 \qquad \text{... (i)}$$

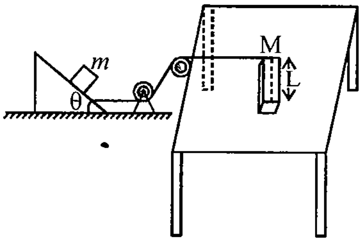
$$\text{and}\quad \frac{T^2Mg}{4\pi^2}\cdot l_2 = I_C + Ml_2^2 \qquad \text{... (ii)}$$

Subtracting (i) and (ii) we get

$$\frac{T^2Mg}{4\pi^2}\cdot(l_1 - l_2) = M(l_1 + l_2)(l_1 - l_2)$$

$$\text{or}\quad \frac{T^2g}{4\pi^2} = L \quad \text{or} \quad T = 2\pi\sqrt{\tfrac{L}{g}}$$

27. Consider the situation shown in the figure below. Table is fixed and the rod lying on the table can rotate about one end which is hinged to the table. Coefficient of static friction between rod and table is $\mu$.

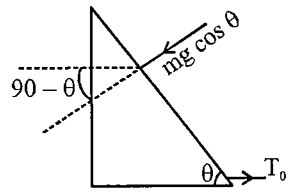


All the strings and pulleys are light and frictionless. There is no friction between the two blocks and between triangular block and ground. What should be maximum value of mass '$m$' so that the triangular block stays in equilibrium.

**Soln.** Let $T_0$ be the maximum tension that can be applied in the string without making the rod rotate.

$\therefore\ T_0 \cdot L$ = torque due to friction

$$= \int_0^L \mu\left(\frac{M}{L}dxg\right)\cdot x = \frac{\mu MgL}{2}$$



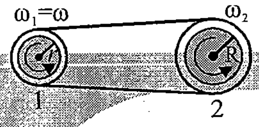
$$\therefore \quad T_0 = \frac{\mu Mg}{2}$$

Now considering the equilibrium of triangular block we get,

$$mg\cos\theta\cos(90-\theta) = T_0$$

$$\therefore \quad m = \frac{\mu M}{\sin 2\theta}$$

28. Two pulleys of radii $r$ and $R$ are connected with a rubber band of weight $w$. If moment of inertia of the two pulleys be $I_1$ and $I_2$ respectively and angular velocity of smaller pulley be $\omega$. Calculate the kinetic energy of the system.



**Soln.** Rubber band has same speed everywhere

$$\therefore \quad \omega_1 r = \omega_2 R \qquad \therefore \quad \omega_2 = \omega\cdot\frac{r}{R}$$

speed of rubber band = $\omega r$

$\therefore$ kinetic energy of the pulley 1 (smaller pulley) $= \frac{1}{2}I_1\omega^2$

$\therefore$ kinetic energy of the pulley 2 (bigger pulley) $= \frac{1}{2}I_2\omega_2^2$

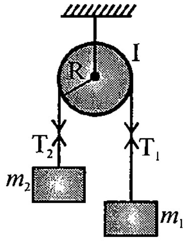
$$= \frac{1}{2}I_2\cdot\omega^2\cdot\frac{r^2}{R^2}$$

Mass of rubber band $= \frac{w}{g}$

$\therefore$ kinetic energy of rubber band $= \frac{1}{2}\left(\frac{w}{g}\right)\cdot\omega^2 r^2$

Total energy of the system $= \frac{1}{2}\left[I_1 + I_2\cdot\frac{r^2}{R^2} + \frac{w}{g}r^2\right]\omega^2$

29. Calculate acceleration of two masses shown in the figure. The strings is light and pulley has moment of inertia $I$ and radius $R$.



**Soln.** Let $T_1$ and $T_2$ be tension in strings connected to $m_1$ and $m_2$ respectively.

$\therefore$ Equation of motion, for $m_1$:

$$m_1 g - T_1 = m_1 a \qquad \text{... (i)}$$

where '$a$' is acceleration (downward) for $m_1$:

$$T_2 - m_2 g = m_2 a \qquad \text{... (ii)}$$

for the pulley:

$$(T_1 - T_2)R = I\alpha$$

$\alpha$ is angular acceleration of pulley.

$$\therefore \quad \alpha = \frac{a}{R} \qquad \therefore \quad T_1 - T_2 = \frac{Ia}{R^2} \qquad \text{... (iii)}$$

Adding equation (i), (ii) and (iii) we get,

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right)a$$

$$\therefore \quad a = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}$$

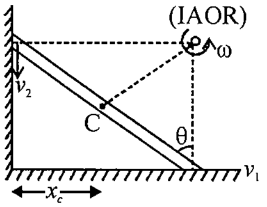
30. A thin rod of length L and mass M lying in vertical position on a horizontal surface and against a smooth vertical wall is displaced slightly due to which its one end starts sliding on the vertical wall. Find angle made with the horizontal when the rod leaves contact with the wall.

**Soln.** Angular velocity of rod is given by $\omega = \dot{\theta}$

where $\theta$ is angle made by rod with vertical. The rod has rotational as well as translational motion. However, motion is pure rotational about the instantaneous axis of rotation (IAOR)

OC = $l/2$

$\therefore$ M.I. about IAOR



$$I_C = \frac{ML^2}{12} + M(L/2)^2 = \frac{ML^2}{3}$$

$\therefore$ from conservation of energy we get,

$$\frac{1}{2} \cdot \frac{ML^2}{3} \cdot \omega^2 = Mg\frac{L}{2}(1 - \cos\theta)$$

$$\frac{\omega^2 L}{3} = g(1-\cos\theta)$$

or $$\frac{L\dot{\theta}^2}{3g} = 1-\cos\theta \qquad \text{... (i)}$$

Distance of centre of rod from the vertical wall

$$x_C = \frac{L}{2}\sin\theta \qquad \therefore\ \dot{x}_C = \frac{L}{2}\cos\theta\dot{\theta}$$

$$\therefore\ \ddot{x}_C = \frac{L}{2}(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$$

when rod leaves the vertical wall, there is no horizontal force on the rod

$$\therefore\ \ddot{x}_C = 0$$

$$\therefore\ \cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2 = 0 \qquad \text{... (ii)}$$

from (i) $\dot{\theta}^2 = \frac{6g}{L}\sin^2\frac{\theta}{2}$

$$\therefore\ \dot{\theta} = \sqrt{\frac{6g}{L}}\cdot\sin\frac{\theta}{2} \quad \text{or} \quad \ddot{\theta} = +\sqrt{\frac{3g}{2L}}\cos\frac{\theta}{2}\cdot\dot{\theta}$$

$\therefore$ from (ii) we get

$$+\cos\theta\cdot\sqrt{\frac{3g}{2L}}\cos\frac{\theta}{2}\dot{\theta} - \sin\theta\dot{\theta}^2 = 0$$

or $$\sqrt{\frac{3g}{2L}}\cos\theta\cdot\cos\frac{\theta}{2} - \sin\theta\cdot\sqrt{\frac{8g}{L}}\cdot\sin\frac{\theta}{2} = 0$$

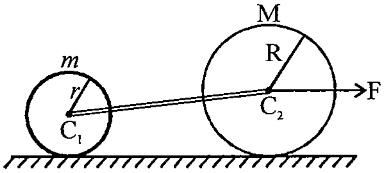
$$\cos\theta\cdot\cos\frac{\theta}{2} - 2\sin\theta\cdot\sin\frac{\theta}{2} = 0$$

$$\frac{\cos\theta\cdot\cos\frac{\theta}{2}}{\sin\theta\cdot\sin\frac{\theta}{2}} = \frac{2}{1} \quad \text{or} \quad \frac{(2\cos^2\frac{\theta}{2} - 1)}{(1-\cos^2\frac{\theta}{2})} = \frac{4}{1}$$

solving we get $\theta = \cos^{-1}\left(\frac{2}{3}\right)$

Therefore, angle made with horizontal $= \frac{\pi}{2} - \cos^{-1}\left(\frac{2}{3}\right) = \sin^{-1}\left(\frac{2}{3}\right)$

31. Two wheels of masses M and $m$ and radii R and $r$ respectively are connected with a rod of length L. The system rolls on a rough surface under the action of an applied force F. Find tension in the connecting rod.



**Soln.** Since the two wheels are connected their linear acceleration must be same.

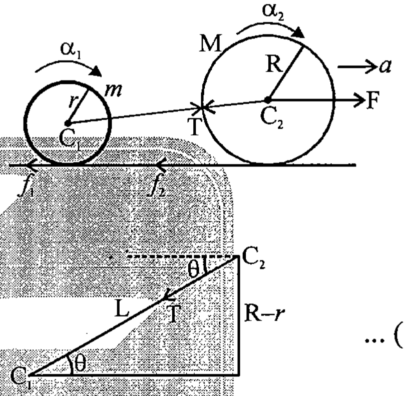
Let T be the tension in the connecting rod.

$\therefore$ Equation of motion for 'M'

$$F - T\cos\theta - f_2 = Ma \qquad \text{... (i)}$$

and $f_2 \cdot R = \dfrac{MR^2}{2} \cdot \alpha_2$

Here $\alpha_2 = \dfrac{a}{R}$ for rolling motion



$\therefore\ f_2 = \dfrac{MR}{2} \cdot \left(\dfrac{a}{R}\right)$

or $f_2 = \dfrac{MR}{2}$ ... (ii)

Equation of motion for '$m$'

$$T\cos\theta - f_1 = ma \qquad \text{... (iii)}$$

and $f_1 \cdot r = \dfrac{mr^2}{2} \cdot \alpha_1$

here $\alpha_1 = \dfrac{a}{r}$ for rolling motion

$\therefore\ f_1 = \dfrac{ma}{2}$ ... (iv)

adding (i), (ii), (iii) and (iv) we get,

$$F = \frac{3}{2}(M+m)a \qquad \therefore \quad a = \frac{2F}{3(M+m)}$$

from equation (iv)

$$f_1 = \frac{ma}{2} = \frac{mF}{3(M+m)}$$

$\therefore$ from (iii) $T\cos\theta = ma + f_1$

$$= \frac{2mF}{3(M+m)} + \frac{mF}{3(M+m)} = \frac{mF}{(M+m)}$$

$$\therefore \; T = \frac{mF}{(M+m)\cos\theta}$$

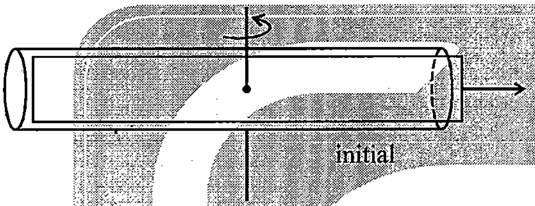
$$T = \left(\frac{m}{M+m}\right)\cdot\frac{F}{\cos\theta}$$

since $\cos\theta = \dfrac{\sqrt{L^2-(R-r)^2}}{L}$

$$\therefore \; T = \frac{m}{M+m}\cdot\frac{FL}{\sqrt{L^2-(R^2-r)^2}}$$

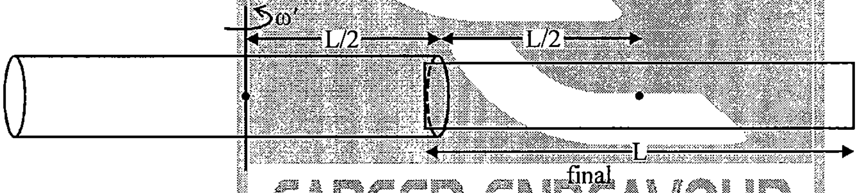
32. A rod of mass M and length L lies inside a smooth tube of same mass and length with their centers slightly off. The tube is rotated with initial angular speed ω about its center. Find the angular speed of the tube when the rod completely comes out of it.

**Soln.** Since, the rod gains radial momentum, angular momentum of system remains conserved.



Initial angular momentum $L_1 = 2\times\dfrac{ML^2}{12}\cdot\omega$

Final angular momentum (when rod has just come out)



$$L_2 = L_2^{rod} + L_2^{tube}$$

$$= \underbrace{\left(\frac{ML^2}{12} + ML^2\right)}_{\text{M.I. of rod about the axis of tube}}\omega' + \frac{ML^2}{12}\omega'$$

$\omega'$ = final angular velocity

$$\because \; L_1 = L_2$$

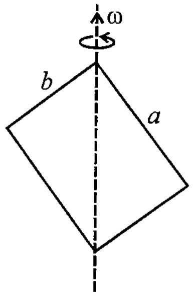
$$\therefore \; \frac{ML^2}{6}\cdot\omega = \frac{7ML^2}{6}\cdot\omega'$$

or $\omega' = \dfrac{\omega}{7}$

33. A thin rectangular plate of mass M and sides '$a$' and '$b$' ($a > b$) is rotating about a vertical axis passing through one of its diagonal. Find the torque required to sustain the rotation.

**Soln.** The rotation axis is not a principal axis, therefore angular momentum is not parallel to $\vec{\omega}$. Let us take $x$ and $y$ axes as shown in the figure.

$$\therefore \; L_x = \frac{Ma^2}{12}\omega_x = \frac{Ma^2}{12}\omega\sin\theta, \quad L_y = \frac{Mb^2}{12}\omega_y = \frac{Mb^2}{12}\cdot\omega\cos\theta$$



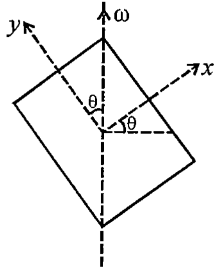
Here $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}, \cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$

Horizontal component of angular momentum

$$L_h = L_x \cos\theta - L_y \sin\theta$$

$$= \frac{M(a^2-b^2)\omega \sin\theta \cdot \cos\theta}{12}$$

$$= \frac{M\omega}{12} \cdot \frac{(a^2-b^2)\cdot ab}{(a^2+b^2)}$$



$L_h$ rotates in horizontal circle with angular velocity $\omega$.

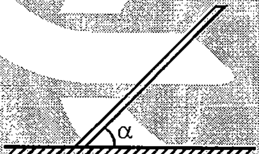
$\therefore$ Torque required to sustain the rotation.

$$\tau = L_h \omega$$

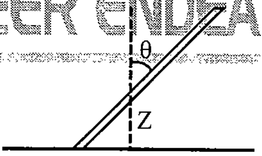
$$= \frac{M\omega^2 \cdot ab}{12} \cdot \left(\frac{a^2-b^2}{a^2+b^2}\right)$$

For a square plate, diagonal is an axis of symmetry. Therefore, no torque is required to sustain the rotation as can be seen by putting $a = b$ in the expression for torque.

34. A uniform thin rod of length L and mass M is held at angle $\alpha$ to a smooth horizontal surface as shown in the figure. The rod is released from this position, find the angular velocity of rod when it becomes horizontal.



**Soln.** Since there is no horizontal force, centre of mass falls straight in vertical line and the lower end of the rod slides on the smooth surface. The rod becomes horizontal when the upper end is about to touch the surface.



Let $\theta$ = angle with the vertical line at time $t$

$\therefore$ angular speed $= \dot{\theta}$

Total K.E. $= \dfrac{1}{2} M \dot{Z}^2 + \dfrac{1}{2} I_c \dot{\theta}^2$

since $Z = \dfrac{L}{2}\cos\theta$

$$\therefore \quad \dot{Z} = -\frac{L}{2}\sin\theta \cdot \dot{\theta}$$

$$\therefore \quad \text{T.K.E.} = \frac{1}{2}\left(ML^2 \frac{\sin^2\theta}{4} + I_c\right)\dot{\theta}^2 = ML^2\left(\frac{\sin^2\theta}{4} + \frac{1}{12}\right)\dot{\theta}^2$$

Loss in P.E. $= Mg\frac{L}{2}(\sin\alpha - \cos\theta)$

$\therefore$ Gain in T.K.E. = loss in P.E.

$$\frac{1}{8}ML^2\left(\sin^2\theta + \frac{1}{3}\right)\dot{\theta}^2 = \frac{ML}{2}g\,(\sin\alpha - \cos\theta)$$

$$\therefore\ \dot{\theta} = \sqrt{4g/L\frac{(\sin\alpha - \cos\theta)}{(\sin^2\theta + 1/3)}} \leftarrow \text{Angular speed at time 't'}$$

When rod becomes horizontal $\theta = 90°$

$$\therefore\ \dot{\theta} = \sqrt{3g/L\sin\alpha} = \sqrt{\frac{3g}{L}\sin\alpha}$$

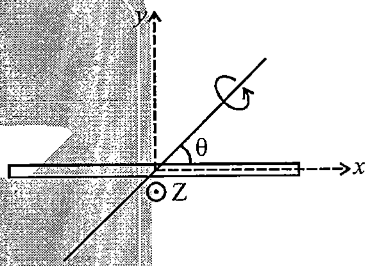
35. Calculate M.I. of a rod about an axis passing through its centre and making an angle θ with it.

**Soln.** This can be easily calculated by integration method. However, we will use a different approach to illustrate the use of inertia tensor.

M.I. about an axis in the direction given by unit vector $\hat{n}$ is given by

$I = \hat{n}\cdot\overrightarrow{\text{II}}\cdot\hat{n}$ ; $\overrightarrow{\text{II}}$ is inertia tensor.

Let us take the co-ordinate system as shown in the figure, therefore.



Here, $\hat{n} \equiv \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

Let L = length and M = mass of the rod.

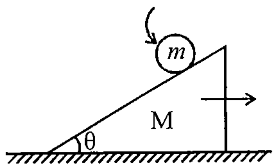
$\therefore\ I_{xx} = 0,\ I_{yy} = I_{zz} = \frac{ML^2}{12}$

As the rod is lying along $x$-axis

$\therefore\ I_{xy} = I_{xz} = f_{y\dot{z}} = 0 \qquad \therefore\ \overrightarrow{\text{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{ML^2}{12} & 0 \\ 0 & 0 & \frac{ML^2}{12} \end{bmatrix}$

$$\therefore\ I = \frac{ML^2}{12}[\cos\theta\ \sin\theta\ 0]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \frac{ML^2}{12}(\cos\theta\ \sin\theta\ 0)\begin{pmatrix} 0 \\ \sin\theta \\ 0 \end{pmatrix} = \frac{ML^2}{12}\sin^2\theta$$

36. A solid sphere rolls on a wedge kept on a smooth surfaces, as shown in the figure, find the acceleration of the wedge.



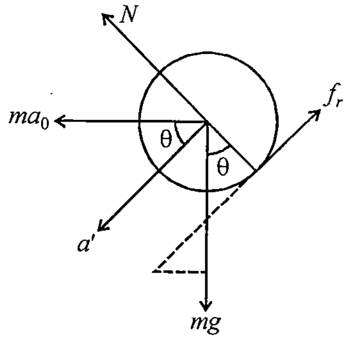
**Soln.** Let $a_0$ is acceleration of wedge and '$a$' is acceleration of solid sphere with respect to the wedge.
Equation of motion of sphere in the frame of wedge

$$mg\sin\theta + m a_0\cos\theta - f_r = ma \qquad ...(1)$$

and $N = m\cdot g\cos\theta - m a_0\sin\theta$ ...(2)

and $f_r\cdot R = I\alpha$

since motion is pure rolling



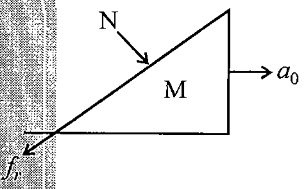
$$\alpha = \frac{a}{R}$$

$$\therefore\ f_r = \frac{Ia}{R^2} \qquad ...(3)$$

From (1) $mg\sin\theta + m a_0\cos\theta = a\left(m + \frac{I}{R^2}\right)$

$$a = \frac{g\sin\theta + a_0\cos\theta}{1 + \frac{I}{mR^2}} \qquad ...(4)$$

Equation of motion of wedge



$$-f_r\cos\theta + N\sin\theta = m a_0$$

or $-\frac{Ia}{R^2}\cdot\cos\theta + N\sin\theta = m a_0$

using (4) and (2) we get,

$$-\frac{I}{R^2}\cos\theta\left(\frac{g\sin\theta + a_0\cos\theta}{1 + \frac{I}{mR^2}}\right) + (mg\cos\theta - m a_0\sin\theta)\sin\theta = m a_0$$

$$\therefore\ -\frac{I}{mR^2}\cdot g\sin\theta\cdot\cos\theta + \left(\frac{I}{mR^2} + 1\right)g\sin\theta\cdot\cos\theta = a_0\left[\frac{M}{m} + \frac{I}{mR^2}\cos^2\theta + \left(1 + \frac{I}{mR^2}\right)\sin^2\theta\right]$$

$$\therefore\ a_0 = \frac{(g\sin\theta\cos\theta)}{\left(\frac{M}{m} + \frac{I}{mR^2} + \sin^2\theta\right)}$$

$$I = \frac{2}{5}mR^2$$

$$\therefore\ a_0 = \frac{g\sin\theta\cdot\cos\theta}{\left(\frac{M}{m} + \frac{2}{5} + \sin^2\theta\right)}$$

37. A cylinder of 1kg mass and 0.02m diameter left at the top of an inclined plane of height 1m rolls down without slipping (Take $g = 9.8\ \text{m/s}^2$). **[GP-2006]**

(a) What is kinetic energy of cylinder when it reaches at the bottom of inclined plane?

(b) Find velocity of centre of mass of cylinder on reaching the bottom of inclined plane.

**Soln.** (a) In rolling motion total mechanical energy remains conserved. Therefore, kinetic energy at the bottom will be equal to potential energy at the top.

$\therefore$ kinetic energy at bottom $= mgh$

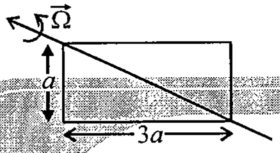
$$= 1 \times 9.8 \times 1 = 9.8 \text{ Joule}$$

(b) In rolling motion, kinetic of a cylinder is $\frac{3}{4}mv_c^2$ where $v_c$ is velocity of centre of mass.

$$\therefore \quad \frac{3}{4}mv_c^2 = 9.8$$

$$\frac{3}{4}\times 1\times v_c^2 = 9.8 \Rightarrow v_c = \sqrt{\frac{9.8\times 4}{3}} = 3.6\text{m/s}$$

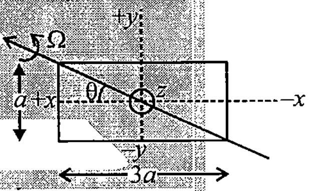
38. (a) A uniform rectangular plate of length $3a$ and width $a$ has mass M. Ignoring gravity, determine the magnitude and direction of the torque $\vec{\tau}$ required to rotate the plate with uniform angular velocity $\vec{\Omega}$ in counter clock-wise direction about the diagonal shown below.



(b) A smaller rectangular plate of length $3a/2$ and width $a/2$ is cut from the plate shown above. Find the torque $\vec{\tau}$ to rotate the new plate along the diagonal with the same angular velocity $\vec{\Omega}$. **[GP-2012]**

**Soln.** Let us choose principal axes ($x$, $y$, $z$) as shown in figure.

Angular momentum of the plate



$$\vec{L} = I_{xx}\omega_x\hat{x} + I_{yy}\omega_y\hat{y} + I_{zz}\omega_z\hat{z}$$

$$= \frac{Ma^2}{12}\Omega\cos\theta\hat{x} + \frac{M(3a)^2}{12}\Omega\sin\theta\hat{y} + \frac{M}{12}[a^2+(3a)^2]\times 0$$

$$\vec{L} = \frac{Ma^2}{12}\Omega\cdot\frac{3}{\sqrt{10}}\hat{x} + \frac{3Ma^2}{4}\Omega\frac{1}{\sqrt{10}}\hat{y} = \frac{3\Omega}{4\sqrt{10}}Ma^2\left(\frac{1}{3}\hat{x}+\hat{y}\right)$$

$$\vec{\Omega} = \Omega(\cos\theta\hat{x} + \sin\theta\hat{y}), \quad \vec{\Omega} = \frac{\Omega}{\sqrt{10}}(3\hat{x}+\hat{y})$$

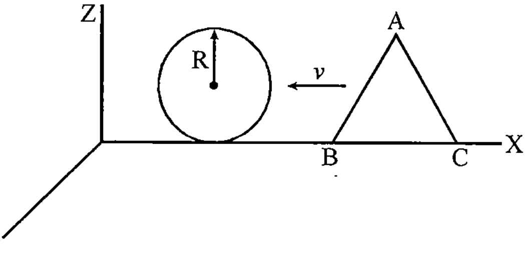
Torque required $\vec{\tau} = \vec{\Omega}\times\vec{L}$

$$= \frac{3Ma^2\Omega^2}{40}(3\hat{x}+\hat{y})\times\left(\frac{1}{3}\hat{x}+\hat{y}\right) = \frac{3Ma^2\Omega^2}{40}\left(3\hat{z}-\frac{1}{3}\hat{z}\right) = \frac{Ma^2\Omega^2\hat{z}}{5}$$

Thus a torque of magnitude $\frac{Ma^2\Omega^2}{5}$ is required in a direction perpendicular to the plane of the plate.

(b) Replacing $a$ by $a/2$ we get $\vec{\tau} = \frac{Ma^2\Omega^2}{20}\hat{z}$.

39. A wedge of mass $m$ and triangular cross section $(AB = BC = CA = 2R)$ is moving with a constant velocity $-v\,i$ toward sphere of radius $R$ fixed on smooth horizontal table as shown in figure. The wedge makes an elastic collision with the fixed sphere and return along the same path without any rotation. Neglect all friction and suppose that the wedge remains in contact with the sphere for a very short time $\Delta t$, during which the sphere exerts a constant force $F$ on the wedge.



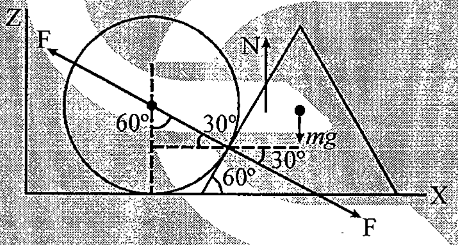
(a) Find the force $F$ and also normal force $N$ exerted by the table on the wedge during the time $\Delta t$.

(b) Let $h$ denote the perpendicular distance between the center of mass of the wedge and the line of action of $F$. Find the magnitude of the torque due to the normal force $N$ about the center of the wedge during the interval $\Delta t$.

**Soln.** (a) The collision is perfectly elastic collision. Hence the wedge will recede with the same velocity as sphere is fixed.

The change in momentum of the wedge is given by $\Delta p_x = 2mv$ ... (i)

At the time of collision, the different forces are shown in figure.



Component of force along X-axis = $F\cos 30$

Hence $F\cos 30 \times \Delta t = 2mv$ or $F = \dfrac{4mv}{\sqrt{3}\,\Delta t}$ ... (ii)

In vector form,

$$F = F\cos 30\, i + F\sin 30\,(-k)$$

or $$F = \frac{4mv}{\sqrt{3}\,\Delta t}\cdot\frac{\sqrt{3}}{2}\,i + \frac{4mv}{\sqrt{3}\,\Delta t}\cdot\frac{1}{2}(-k) = \frac{2mv}{\Delta t}\left[i - \frac{1}{\sqrt{3}}\cdot k\right] \quad \text{... (iii)}$$

For vertical direction

$$mg + F\sin 30 = N$$

$$mg + \frac{4mv}{\sqrt{3}\cdot\Delta t}\cdot\frac{1}{2} = N$$

$$\therefore \quad N = \frac{2mv}{\sqrt{3}\cdot\Delta t} + mg \quad \text{... (iv)}$$

In vector form, $N = -\left[2mv/\sqrt{3}\cdot\Delta t + mg\right]k$ ... (v)

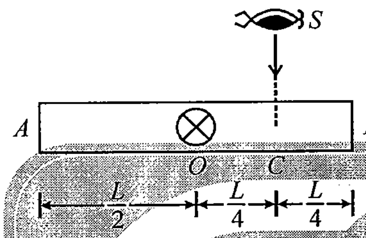
(b) Taking the moments about the center of the wedge

$$\sum \vec{\tau} = 0 \text{ i.e., } \overrightarrow{\tau_F} + \overrightarrow{\tau_N} + \overrightarrow{\tau_m} = 0$$

Here $\overrightarrow{\tau_{mg}} = 0, \therefore \overrightarrow{\tau_N} = -\overrightarrow{\tau_F}$ or $\overrightarrow{\tau_N} = F \cdot hj = \frac{4mvh}{\sqrt{3}\cdot \Delta t}\cdot j$

Magnitude of $\tau_N = \frac{4mvh}{\sqrt{3}\,\Delta t}$ ... (vi)

40. A homogeneous rod $AB$ of length $L = 1.8$ m and mass $M$ is pivoted at the center $O$ in such a way that it can rotate freely in the vertical plane (see figure). The rod is initially in the horizontal position. An insect $S$ of the same mass $M$ falls vertically with speed $v$ on the point $C$, midway between the points $O$ and $B$. Immediately after falling, the insect moves towards the end $B$ such that the rod rotates with a constant angular velocity $\omega$.



(a) Determine the angular velocity $\omega$ in terms of $v$ and $L$.
(b) If the insect reaches the end $B$ when the rod has turned through an angle of 90°, determine $v$.

**Soln.** (a) The initial angular momentum of the insect about $O = (M\,v)\,(L/4)$.

Moment of inertia of the rod about an axis passing through $O$ and perpendicular to the rod $= (ML^2/12)$.

The moment of inertia of the insect when at $C = M\,(L/4)^2$.

Therefore, moment of inertia of the rod plus insect when at $C$.

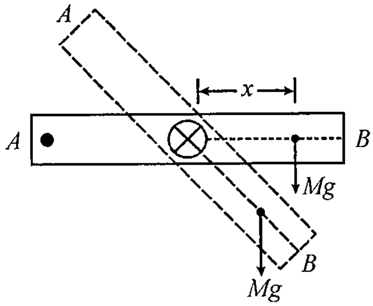
$$= \frac{ML^2}{12} + M\left(\frac{L}{4}\right)^2 = \frac{ML^2}{12} + \frac{ML^2}{16}.$$

Angular momentum in this case $= \left(\frac{ML^2}{12} + \frac{ML^2}{16}\right)\omega$.

From the law of conservation of angular momentum $\frac{MvL}{4} = \left(\frac{ML^2}{12} + \frac{ML^2}{16}\right)\omega$

Therefore, $\omega = \frac{12v}{7L}$.

(b) After falling, the insect moves towards the end $B$. Now the moment of inertia of the system changes. Let the insect be at a distance $x$ after time $t$ as shown in figure.



At this instant, the moment of inertia of the system $= \left(\frac{ML^2}{12}+Mx^2\right)$

Angular momentum $= \left(\frac{ML^2}{12}+Mx^2\right)\omega$

Torque $\tau =$ rate of change of angular momentum $= \frac{d}{dt}\left[\frac{ML^2}{12}+Mx^2\right]\omega = M\omega 2x\frac{dx}{dt}$ ($\because$ $\omega$ is constant).

The torque is provided by the force $Mg$ acting downward i.e., $\tau = Mgx$

Therefore, $Mgx = M\omega 2x\frac{dx}{dt}$ or $\frac{dx}{dt}=\frac{g}{2\omega}$

Therefore, velocity of insect $v=\frac{dx}{dt}=\frac{g}{2\omega}$

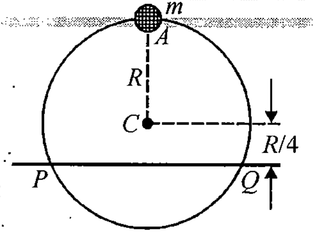
Time taken to rotate through $\frac{\pi}{2}=\frac{\pi}{2\omega}$

Time taken by insect to move to the other end of the rod $=\frac{L/4}{v}=\frac{L}{4}\times\frac{2\omega}{g}=\frac{\omega L}{2g}$

Equating the two time periods, we have $\frac{\pi}{2\omega}=\frac{\omega L}{2g}$ or $\omega^2 L=\pi g$ or $\omega=\sqrt{\left(\frac{\pi g}{L}\right)}$.

$$\frac{12v}{7L}=\sqrt{\left(\frac{\pi g}{L}\right)} \Rightarrow v=\frac{7}{12}L\sqrt{\left(\frac{\pi g}{L}\right)}=\frac{7}{12}\sqrt{(\pi g L)}=\frac{7}{12}\sqrt{(3.14\times10\times1.8)}=\mathbf{4.4\ m/s}.$$

41. A uniform circular disc has radius $R$ and mass $m$. A particle also of mass $m$, is fixed at point $A$ on the edge of the disc as shown in figure. The disc can rotate freely about a fixed horizontal chord $PQ$ that is at a distance $R/4$ from the center $C$ at the disc. The line $AC$ is perpendicular to $PQ$.



Initially, the disc is held vertical with the point $A$ at its highest position. It is then allowed to fall so that it starts rotating about $PQ$. Find the linear speed of the particle as it reaches the lowest position.

**Soln.** First of all we shall calculate the moment of inertia of the disc about the axis $PQ$. We know that the moment of inertia of a disc about any diameter is $(mR^2/4)$. Applying the theorem of parallel axis, the moment of inertia of the disc about $PQ$ is given by

$$(I_{\text{disc}})_{PQ}=\left[\frac{mR^2}{4}+m\left(\frac{R}{4}\right)^2\right]=\frac{5}{16}mR^2 \qquad \text{...(i)}$$

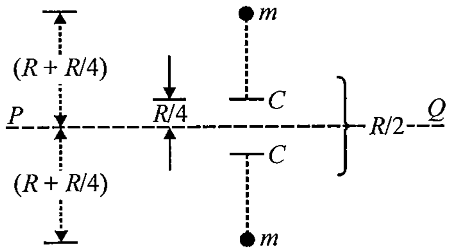
The moment of inertia of mass $m$ about PQ is given by

$$(I_{\text{particle}})_{PQ} = m\left(R+\frac{R}{4}\right)^2 = m\left(\frac{5R}{4}\right)^2 = \frac{25\,mR^2}{16} \qquad \text{... (ii)}$$

So, the moment of inertia of the system (disc + particle) about $PQ$ is given by

$$I = \frac{5}{16}mR^2 + \frac{25}{16}mR^2 = \frac{30\,mR^2}{16} = \frac{15\,mR^2}{8} \qquad \text{... (iii)}$$

Now we shall calculate the loss in P.E. when the particle reaches in lowest position. The lowest position of particle is shown in figure.



Loss in potential energy of disc $= mg\left(\frac{R}{4}+\frac{R}{4}\right) = \frac{mgR}{2}.$

Loss in potential energy of particle $= mg\left[\left(R+\frac{R}{4}\right)+\left(R+\frac{R}{4}\right)\right] = \frac{5\,mgR}{2}.$

Total loss in potential energy $= \frac{mgR}{2} + \frac{5\,mgR}{2} = 3\,mgR.$

Let $\omega$ be the angular velocity of the disc at the lowest position. So,

$$\frac{1}{2}I\,\omega^2 = \text{Loss in potential energy} \quad \text{or} \quad \frac{1}{2}\times\frac{15\,mR^2}{8}\times\omega^2 = 3\,mgR.$$

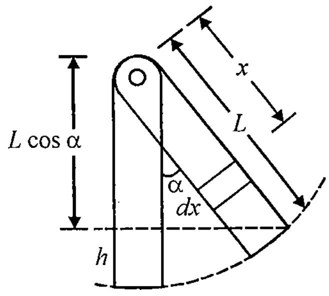
Solving, we get $\omega = \sqrt{\left(\frac{16\,g}{5R}\right)}$.

Hence, the linear speed of the particle is given by

$$v = \left(R+\frac{R}{4}\right)\omega = \frac{5R}{4}\times\sqrt{\left(\frac{16g}{5R}\right)} = \sqrt{(5gR)}.$$

42. One end of a uniform rod of mass $M$ and length $L$ is supported by a frictionless hinge which can withstand a tension of $1.75\,Mg$. The rod is free to rotate in a vertical plane. To what maximum angle should the rod be rotated from the vertical position so that when left, the hinge does not break.

**Soln.** See figure



From figure, $h = L - L\cos\alpha = L(1-\cos\alpha)$

Rotational kinetic energy $= \frac{1}{2}I\omega^2$, where $I = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}$.

According to the law of conservation of energy

$$\frac{1}{2}I\omega^2 = \frac{1}{2}MgL(1-\cos\alpha)$$

or $$\frac{1}{2}\left(\frac{ML^2}{3}\right)\omega^2 = \frac{1}{2}MgL(1-\cos\alpha)$$

or $$\omega^2 = \frac{3g}{L}(1-\cos\alpha) \qquad \text{... (i)}$$

Consider an element of length $dx$ of the rod at a distance $x$ from hinge. Centrifugal force on this element

$$dF = \frac{M}{L}x\,dx\,\omega^2 \text{ or } F = \int_0^L dF = \frac{ML\omega^2}{2} \qquad \text{... (ii)}$$

Substituting the value of $\omega^2$ from equation (i) in equation (ii), we get

$$F = \frac{3}{2}Mg(1-\cos\alpha) \qquad \text{... (iii)}$$

Now, total downward force will be $(F + Mg)$. Hence according to the given problem

$$F + Mg = 1.75\,Mg \text{ or } F = 0.75\,Mg \qquad \text{... (iv)}$$

From equation (iii) and (iv), we get

$$0.75\,Mg = \frac{3}{2}Mg(1-\cos\alpha)$$

or $$(1-\cos\alpha) = \frac{1}{2} \text{ or } \cos\alpha = \frac{1}{2}$$

$\therefore$ $\alpha = \mathbf{60^\circ}$.

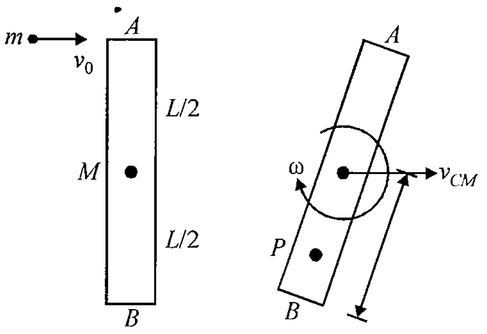
43. A rod $AB$ of mass $M$ and length $L$ is lying on a horizontal frictionless surface. A particle of mass $m$ travelling along the surface hits the end '$A$' of the rod with a velocity $v_0$ in a direction perpendicular to $AB$. The collision is completely elastic. After the collision the particle comes to rest.

(a) Find the ratio $m/M$

(b) $A$ point $P$ on the rod is at rest immediately after the collision. Find the distance $AP$

(c) Find the linear speed of the point $P$ at a time $\frac{\pi L}{(3v_0)}$ after the collision

**Soln.** (a) See figure



Applying the conservation of momentum $m v_0 = M v_{CM}$

$$\therefore \qquad v_{CM} = \frac{m v_0}{M} \qquad \text{... (i)}$$

Applying the conservation of angular momentum

$$m v_0 \left(\frac{L}{2}\right) = \left(\frac{ML^2}{12}\right)\omega \quad \text{or} \quad \omega = \frac{6 m v_0}{ML} \qquad \text{... (ii)}$$

Applying the conservation of kinetic energy, we get

$$\frac{1}{2} m v_0^2 = \frac{1}{2} M v_{CM}^2 + \frac{1}{2}\left(\frac{ML^2}{12}\right)\omega^2 \qquad \text{... (iii)}$$

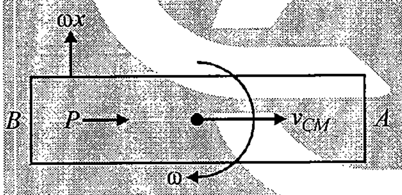
Substituting the values of $v_{CM}$ and $\omega$ from equation (i) and (ii) in equation (iii) and solving we have

$$\frac{m}{M} = \frac{1}{4} \qquad \text{... (a)}$$

(b) Let the point $P$ be at a distance of $x$ from center of rod.

$$\therefore \qquad v_{CM} - \omega x = 0 \quad \text{or} \quad x = \left(\frac{v_{CM}}{\omega}\right)$$

$$\therefore \qquad x = \frac{(m v_0/M)}{(6 m v_0/ML)} = \frac{L}{6} \qquad \text{... (b)}$$



(c) After time $t = \left(\frac{\pi L}{3 v_0}\right)$, the angle traversed $\theta$ is given by $\theta = \omega t = \frac{6 m v_0}{ML} \times \frac{\pi L}{3 v_0} = \frac{\pi}{2}$ radian.

Net velocity of $P$ at this instant $v_P = \sqrt{(v_{CM})^2 + \omega^2 x^2}$

Substituting the values and solving, we get $v_P = \frac{v_0\sqrt{2}}{4}$.

# Chapter 10

# *Fluid Dynamics*

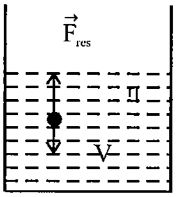
Viscosity is the property of fluid due to which fluid flow is apposed.

**Frictional in nature:** It is typically proportional to the velocity and non-conserved in nature.

$$F_{\text{resistance}} \propto (-V)$$

$\vec{F} = -bV$ where B = demping constant medium property.

In case of fluids, $b = \eta \rightarrow$ coefficient of viscosity.



Stokes law

$$\vec{F}_{\text{resistance}} = k\eta\vec{V} \qquad \left(f = \eta\frac{dv}{dx}, \text{ viscous drag}\right)$$

**Dimension analysis:**

F depends on $\gamma, \eta, v$ we write

$$F = k\,\gamma^x\,\eta^y\,v^z$$

$$\left(MLT^{-2}\right) = \left(ML^{-1}T^{-1}\right)^y \left(LT^{-1}\right)^z L_x = M^y L^{x-y+z} T^{-y-z}$$

**Dimension of η**

$$\eta = \frac{f}{\frac{dv}{dx}} = \frac{F/A}{\frac{dv}{dx}} = \frac{MLT^{-2}/L^2}{T^{-1}} = \boxed{ML^{-1}T^{-1}}$$

Here, $y = 1$

$(y+z) = 2 \qquad z = 1\ (y+z) = 2 \qquad z = 1$

$x - y + z = 1 \qquad x = 1\ x - y + z = 1 \qquad x = 1$

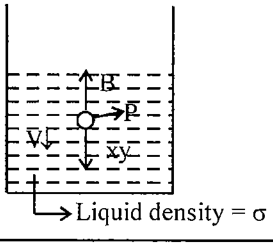
$\vec{F} = k\eta\,r\,\vec{v}$

$k = 6\pi$ experimentally.

Then, $F = 6\pi\eta r\vec{v}$ ( r = radius of the particle)

**Application of Stokes Law:**

$m = \frac{4}{3}\pi r^3 \rho$, $\pi$ = density of the ball



Buoyant force $= \left(\frac{4}{3}\pi r^3\sigma\right)g$

Equation of motion, $m\frac{d\vec{v}}{dt} = \frac{4}{3}\pi r^3(\rho-\sigma)g - 6\pi\eta r\vec{v},\ \frac{4}{3}\pi r^3\rho\frac{d\vec{v}}{dt}$

Or, $$\frac{d\vec{v}}{dt} = \left(1-\frac{\sigma}{\rho}\right)g - \frac{9}{2}\frac{\eta}{\rho}\frac{\vec{v}}{r^2}$$

If $F_{net} = 0,\ \frac{d\vec{v}}{dt} = 0,\ \vec{v} = \text{constant}$ (terminal velocity)

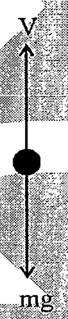
Then, $$\left|\vec{V}_T\right| = \frac{2}{9}\frac{r^2}{\eta}(\rho-\sigma)g$$

$\rho > \sigma \rightarrow$ downward motion

$\rho < \sigma \rightarrow$ upward motion

**General Motion:**

Resistive force $= -bV$



$$m\frac{d\vec{V}}{dt} = \frac{4}{3}\pi r^3(\rho_{air} - \rho_{ball})g - bV$$

$$\frac{4}{3}\pi r^3\rho_{ball}\left(\frac{d\vec{v}}{dt}\right)$$

Or, $$\frac{d\vec{v}}{dt} = g\left(\frac{\rho_{air}}{\rho ball} - 1\right) - \frac{bV}{\frac{4}{3}\pi r^3\rho}$$

Or, $$\frac{d\vec{v}}{dt} = -g\left(1-\frac{\rho_{air}}{\rho_{ball}}\right) - \frac{b}{\frac{4}{3}\pi r^3\rho_{ball}}$$

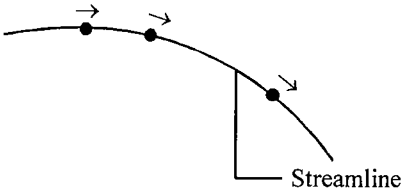
Take, $$g\left(1-\frac{\rho_{air}}{\rho_{ball}}\right) = C$$

And $$\frac{b}{\frac{4}{3}\pi r^3\tau_{ball}} = \frac{b}{m};\ \frac{dv}{dt} = -\left(c + \frac{b}{mr}\right)$$

Or, $$\int_{v_0}^{v}\frac{dv}{c+\frac{b}{mv}} = -c\int_0^t dt\ ;\ \left(\frac{m}{b}\right)\ell n\left(c+\frac{b}{m}\right)\Bigg|_{v_0}^{v} = -ct \qquad \Rightarrow \left(\frac{m}{b}\right)\ell n\left(\frac{c+\frac{b}{m}}{c+\frac{b}{m}v_0}\right) = -ct$$

$$\Rightarrow \quad \frac{c+\frac{b}{m}v}{c+\frac{b}{m}v_0} = \exp-\left(\frac{cb}{m}\right)t \quad \Rightarrow v = \left(\frac{m}{b}\right)\left(c+\frac{b}{m}v_0\right)e^{-\frac{cb}{m}t} - \frac{cm}{b}$$

**Sreamline (Laminar) flow:** Flow where are paticles (preceding or succeeding) have same velocity at a particle point.



The path followed by the particles is called steamline.

Velocity at any point of streamline is tangential to the streamline at that point.

No two streamline cross each other.

**Turbulant:** Which is not streamline. Velocity of particles at a given point varies with time both in magnitude and direction in an irregular manner.

The streamline flow is maintained when the velocity is below a particular limit called critical velocity.

Critical velocity $V_c$ is related to the density of the liquid $\rho$, coefficient of viscosity $(\eta)$ and the radius of the tube.

$$R = \frac{\text{ratio of inertial force}}{\text{viscous force}} = \left(\frac{\rho V_c r}{\eta}\right)$$

$R < 2000 =$ streamline
$2000 < R < 3000 =$ transition
$R > 3000 =$ turbulant.

**Dimension:** $v = k\eta^x \rho^y r^z;\ LT^{-1} = k\left(ML^{-1}T^{-1}\right)^x \left(ML^{3}\right)^y L^z = k\,M^{x+y}L^{-x-3y+z}T^{-x};\ x+y=0$

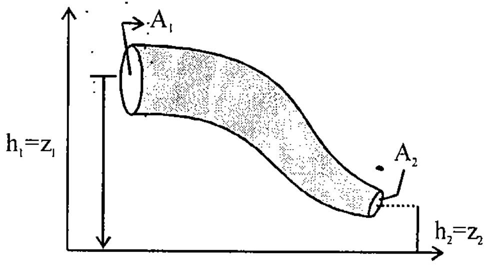
$-x=-1,\ x=1,\ y=-x=-1 \qquad (x=1;\ y=z=-1)$

$-x-3y+z=1,\ \ -1+3+z=1,\ z=-1$

Hence, $v = \left(\frac{k\eta}{\rho r}\right)$

**Bernoulli's equation:** Ideal fluid flow

(i) Streamline (ii) Incompressible (iii) Non viscous



Mass entering through $A_1$

$$\Delta m = \rho A_1 \Delta x_1$$

$$\lim_{\Delta t\to 0}\frac{\Delta m}{\Delta t} = \frac{dm}{dt} = \rho A_1 v_1 \qquad \text{Rate of mass entering.}$$

Rate of mass coming out $= \dfrac{dm}{dt} = \rho A_1 v_2$

As the fluid is incompressible

$$\rho A_1 v_1 = \rho A_2 v_2 \Rightarrow \boxed{A_1 v_1 = A_2 v_2}\text{; General continuity equation, } \vec{\nabla}.(\rho\vec{v}) = 0$$

Now let work done = change in mechanical energy

Work done on fluid element of mass $\Delta m$ by the pressure force is

where, $\Delta x$ = displacement

$$(P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t) = \Delta m\left[\frac{P_1}{\rho} - \frac{P_2}{\rho}\right] = (P_1 - P_2)\frac{\Delta m}{\rho}$$

$$\Delta m = \rho_1 v_1 \Delta t = \rho A_2 v_2 \Delta t$$

Net change in mechanical energy

$$\left(\frac{1}{2}\Delta m v_2^2 + \Delta m g z_2\right) - \left(\frac{1}{2}\Delta m v_1^2 + \Delta m g z_1\right)$$

Equating with work done

$$(P_1 - P_2)\frac{\Delta m}{\rho} = \frac{1}{2}\Delta m v_2^2 + \Delta m g z_2 - \frac{1}{2}\Delta m v_1^2 - \Delta m g z_1$$

$$\Rightarrow \quad P_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2 = P_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1$$

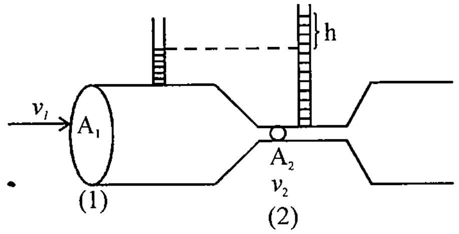
i.e. $$\boxed{\left(P + \frac{1}{2}\rho v^2 + \rho g z\right) = \text{constant}}$$

Or, $$\boxed{\frac{P}{\rho g} + \frac{v^2}{2g} + z = \text{constant}}$$

or, $$\boxed{PV + \frac{1}{2}mv^2 + mgz = \text{constant}}$$

**Venturimeter:** Measures the flow

Both point have same height.



$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \;\;\&\;\; A_1 v_1 = A_2 v_2$$

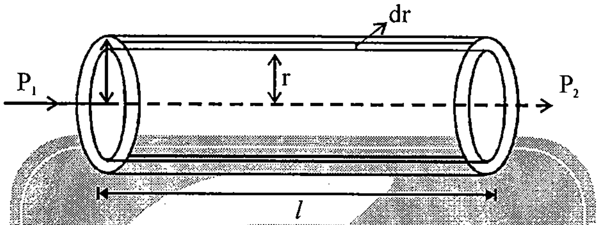
$$(P_2 - P_1) = \frac{1}{2}\rho\left(v_1^2 - v_2^2\right)$$

Or, $$\rho gh = \frac{1}{2}\rho\left(v_1^2 - \left(\frac{A_1}{A_2}v_1\right)^2\right)$$

Or, $$v_1^2 = \frac{2ghA_2^2}{A_2^2 - A_1^2}$$

Flow, $$Q = v_1A_1 = \left(\frac{A_1A_2\sqrt{2g}}{\sqrt{A_1^2 - A_2^2}}\right)\sqrt{h} = k\sqrt{h}$$

**Poiseulle's formula:**



**Assumption:**

(1) No radial flow only streamline flow due to constant pressure difference.

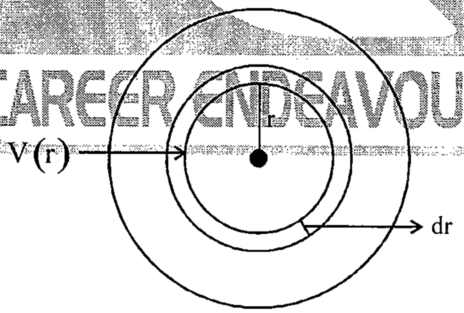
(2) Uniform cross section of the pipe.

(3) Fluid in contact with walls of the pipe is at rest.

Lets take a pipe of uniform cross section of length '$l$' and radius 'a', pressure at entering and leaving point of pipe is $P_1$ and $P_2$ respectively.

Velocity of the liquid decreases gradually from the axis to the wall where it is zero.

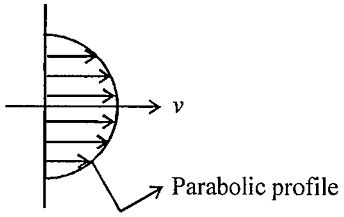
Take $V = V(r)$ at some distance 'r' from the axis.



Viscous drag/unit area $= \eta\frac{dv}{dr}$. Total resisting force over the surface of cylinder of radius 'r' is $\eta\frac{dv}{dr}A$

$$= 2\pi r\ell\eta\frac{dv}{dr}$$

$$P_1 - P_2 = P$$



Now pressure difference = P

Active force due to pressure difference = $P\pi r^2$

$$\therefore \quad P\pi r^2 = -\eta 2\pi r\ell\frac{dv}{dr} \qquad \left[\frac{dv}{dt} = -\text{ve as v decreases with increasing r}\right]$$

Or, $$Pr = -2\pi\ell\frac{dv}{dr}$$

Or, $$\int_{v(r)}^{0} dv = -\frac{P}{2\eta\ell}\int_{r}^{a} r\,der\,;\ -V(r) = -\frac{P}{4\eta\ell}\left(a^2 - r^2\right);\ V(r) = \frac{P}{4\eta\ell}\left(a^2 - r^2\right)$$

Or, $$V(0) = \frac{Pa^2}{4\eta\ell},\ V(a) = 0$$

Now if dv is the volume passed through cylinderical shell of radius 'r' and 'r+dr' then

$$dv = 2\pi r dr v \qquad (Av)$$

$$= \frac{\pi P}{2\eta\ell}\left(a^2 - v^2\right) r dr$$

Total volume passed through the tube

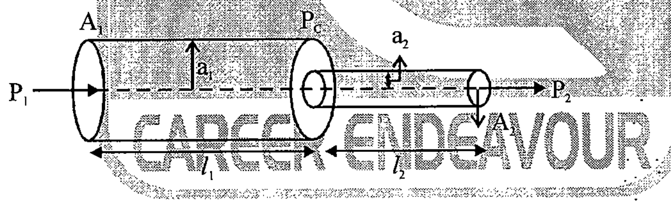
$$= \frac{\pi P}{2\eta\ell}\int_0^a \left(a^2 - r^2\right) = \frac{\pi P}{2\eta\ell}\left(\frac{a^2r^2}{2} - \frac{r^4}{4}\right)_0^a = \frac{\pi P}{2\eta\ell}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$\boxed{Q = \frac{\pi P a^4}{8\eta\ell}}$$

**Applications:** Joining of pipes or capilaries (series or parallel)

$$\begin{bmatrix} \text{Voltage} \Leftrightarrow \text{pressure (P)} \\ \text{Current} \Leftrightarrow \text{Flow(Q)} \end{bmatrix}$$

**1. Pipes in series:**

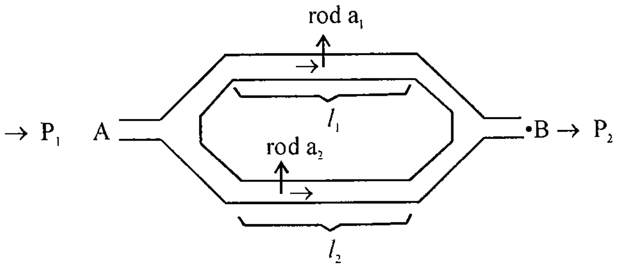


$(P_1 - P_2)$ = Net pressure difference.

$$Q_1 = \frac{\pi(P_1 - P_C)a_1^4}{\rho\eta\ell_1} \qquad Q_1 = Q_2 \Rightarrow \frac{(P_1 - P_C)a_1^4}{\ell_1} = \frac{(P_C - P_2)a_2^4}{\ell_2}$$

$$Q_2 = \frac{\pi(P_C - P_2)a_2^4}{\rho\eta\ell_2} \qquad \Rightarrow P_C = \left(\frac{P_1a_1^4}{\ell_1} + \frac{P_2a_2^4}{\ell_2}\right)\frac{1}{\left(\frac{a_1^4}{\ell_1} + \frac{a_2^4}{\ell_2}\right)}$$

**Pipes in parallel:**



$\Delta P$ same for both; $Q_1 \neq Q_2$; $Q = Q_1 + Q_2$

$$Q_1 = \frac{\pi P a_1^4}{8\eta \ell_1} \qquad Q_2 = \frac{\pi P a_2^4}{8\eta \ell_2}; \; Q = Q_1 + Q_2 = \frac{\pi P}{8\eta}\left(\frac{a_1^4}{\ell_1} + \frac{a_2^4}{\ell_2}\right)$$

Put, $\quad R = \frac{8\eta\ell}{\pi a^4} \qquad \Rightarrow V = \frac{P}{R}$

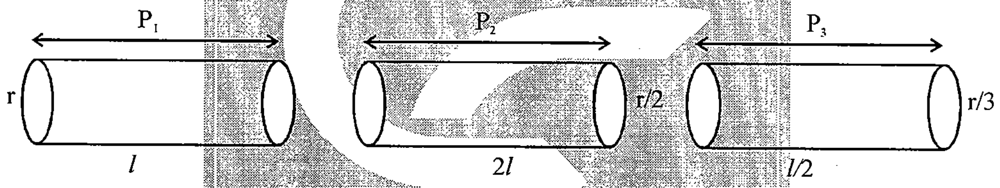
**For parallel,** $\quad V = P\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

**For series,** $\quad P_1 - P_C = Q8\eta\frac{\ell_1}{\pi a_1^4}, \; P_C - P_2 = \frac{Q8\eta\ell_2}{\pi a_2^4}$

$$P_1 - P_2 = \frac{Q8\eta}{\pi}\left(\frac{\ell_1}{a_1^4} + \frac{\ell_2}{a_2^4}\right)$$

Or, $\quad Q = \frac{\pi P}{8\eta} \Big/ \left(\frac{\ell_1}{a_1^4} + \frac{\ell_2}{a_2^4}\right) = P/(R_1 + R_2)$

**Example:**



**Connected in series:**

$$Q = Q_1 = Q_2 = Q_3 = \frac{\pi P_1 r^4}{8\eta\ell} = \frac{\pi P_2 \left(\frac{r}{2}\right)^4}{8\eta(x)} = \frac{\pi P_3 \left(\frac{r}{3}\right)^4}{8\eta \ell/2}$$

Or, $\quad \frac{\pi P_1 r^4}{8\eta\ell} = \frac{\pi P_2 r^4}{32\eta\ell} = \frac{\pi P_3 r^4}{8\eta\ell}\left(\frac{2}{81}\right)$

Or, $\quad P_2 = 32P_1; \; P_3 = \frac{81}{2}P_1$

Net pressure = $P = P_1 + P_2 + P_3 = (1 + 32 + 81)\, P_1$.

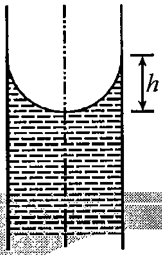
## SOLVED PROBLEMS

1. A liquid is kept in a cylindrical vessel which is rotated along its axis. The liquid rises at the sides. If the radius of the vessel is 0.05 m and the speed of rotation is 2 rev. per sec., find the difference in the height of the liquid at the center of the vessel and at its sides.

**Soln.** When the cylindrical vessel rotates, the velocity at the sides is higher. According to Bernoulli's theorem

$$P + \frac{1}{2}\rho v^2 = \text{constant.}$$

Hence, the pressure at sides is lower. Since the pressure at a given horizontal level must be equal, the liquid rises at sides to difference in the height of the liquid at the center of the vessel and at its sides as shown in figure.



Now, $\frac{1}{2}\rho v^2 = \rho g h$ ... (i)

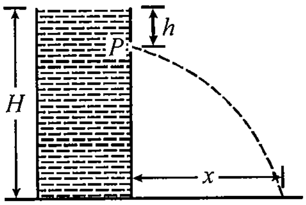
Further, $v = \omega r = 2\pi n \times r = 2\pi \times 2 \times 0.05$ ... (ii)

From equation (i), we get $h = \frac{v^2}{2g}$

Substituting the values, we have $h = \frac{(2\pi \times 2 \times 0.05)^2}{2 \times 9.8} = \mathbf{0.02\ m.}$

2. Water stands at a depth $H$ in a tank, whose side walls are vertical. A hole is made on one of the walls at a depth $h$ below the water surface. Find at what distance from the foot of the wall does the emerging stream of water strike the floor and for what value of $h$ this range is maximum?

**Soln.** The situation is shown in figure.



Here, we have $v_A = \sqrt{(2gh)}$, ... (i)

and $(H - h) = \frac{1}{2}gt^2$ ... (ii)

The distance $x$ is given by $x = v_A \times t$ ... (iii)

From equation (ii), $t = \sqrt{\left(\frac{2(H-h)}{g}\right)}$ ... (iv)

Substituting the value of $v_A$ from equation (i) and the value of $t$ from equation (iv) in equation (iii), we get

$$x=\sqrt{(2gh)}\times\sqrt{[2(H-h)/g}=2\sqrt{[h(H-h)]}.$$

The range $x$ will be maximum when $\dfrac{dx}{dh}=0$

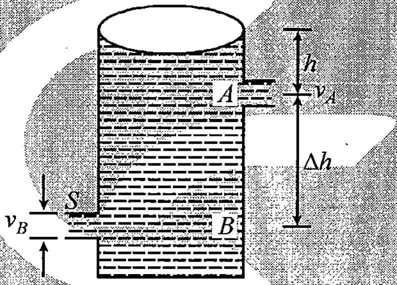
$\therefore \quad 2\cdot\dfrac{1}{2}h^{-1/2}(H-h)^{-1/2}-2h^{1/2}\cdot\dfrac{1}{2}(H-h)^{1/2}=0$

Solving we get $h=\dfrac{H}{2}$.

$$x=2\sqrt{\frac{H}{2}\times\left(H-\frac{H}{2}\right)}=H.$$

3. On the opposite sides of a wide vertical vessel filled with water two identical holes are opened, each having cross-sectional area $S$. The height difference between them is equal to $\Delta h$. Find the resultant force of reaction of the water flowing out of the vessel.

**Soln.** The situation is shown in figure.



Discharge per sec. through $A=S\,v_A$

Discharge per sec. through $B=S\,v_B$

Force of reaction at $A=\rho Q_A v_A=\rho S v_A^2$

Similarly, force of reaction at $B=\rho S v_B^2$

Net force $F=\rho S(v_A^2-v_B^2)$ ... (i)

Applying Bernoulli's theorem at point $A$ and $B$, we have

$$P_a+h\rho g+\frac{1}{2}\rho v_A^2=P_a+(h+\Delta h)\rho g+\frac{1}{2}\rho v_B^2$$

Here $P_a$ is atmosperic pressure $\dfrac{1}{2}\rho\left[v_A^2-v_B^2\right]=\Delta h\,\rho\,g$ ... (ii)

From equation (i) $(v_A^2-v_B^2)=\left(\dfrac{F}{\rho S}\right)$

Substituting this value in equation (ii), we get $\dfrac{1}{2}\rho\left(\dfrac{F}{\rho S}\right)=\Delta h\,\rho\,g$

$\therefore \quad F=(2\,\Delta h\,\rho\,g\,S)$.

4. Water is flowing through two horizontal pipes of different diameters which are connected together. In the first pipe the speed of water is 4 m/sec and the pressure is $2.0 \times 10^4$ n/m$^2$. Calculate the speed and pressure of water in the second pipe. The diameters of the pipes are 3 cm and 6 cm respectively.

**Soln.** From continuity equation, $A_1 v_1 = A_2 v_2$ or $(\pi r_1^2) v_1 = (\pi r_2^2) v_2$

Therefore, $v_2 = \left(\frac{r_1}{r_2}\right)^2 v_1$. Here, $r = 1.5 \text{ cm} = 1.5 \times 10^{-2}$ meter, $r_2 = 3 \times 10^{-2}$ meter and $v_1 = 4$ m/sec.

Therefore, $v_2 = \left(\frac{1.5\times10^{-2}}{3\times10^{-2}}\right) \times 4 = \mathbf{1\ m/sec.}$

Now, by Bernoulli's theorem, $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$.

Therefore, $P_2 = P_1 + \frac{1}{2}\rho (v_1^2 - v_2^2)$

Substituting the given values, we get

$$P_2 = 2.0\times10^4 + \frac{1}{2}\times(1\times10^3)\times(4^2-1^2) = 2.0\times10^4 + 0.75\times10^4 = \mathbf{2.75\times10^4\ newton/meter^2}.$$

5. In a horizontal pipe line of uniform area of cross section, the pressure falls by 8 N-m$^2$ between two points separated by a distance of 1 km. What is the change in kinetic energy per kg of the oil flowing at these points? Density of oil is 806 kg-m$^{-3}$.

**Soln.** According to Bernoulli's theorem, $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$ (pipe horizontal)

or $\quad P_1 - P_2 = \frac{1}{2}\rho (v_2^2 - v_1^2)$

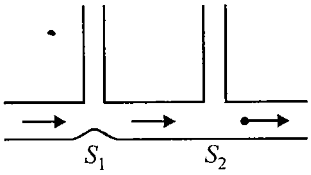
or $\quad \frac{P_1 - P_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2)$

$= $ change in kinetic energy per kg mass.

Therefore, change in kinetic energy per kg mass of oil $= \frac{P_1 - P_2}{\rho}$

Substituting the given values, we have change in K.E. per kg mass $= \frac{8}{800} = \mathbf{10^{-2}\ J/kg.}$

6. Two manometric tubes are mounted on a horizontal pipe of varying cross-section at the section $S_1$ and $S_2$ as shown in figure.



Find the volume of water flowing across the pipe's section per unit time if the difference in water columns is equal to $\Delta h$.

**Soln.** Given that the difference in water columns is equal to $\Delta h$, hence

$$P_1 - P_2 = \Delta h \rho g \text{ or } P_2 - P_1 = \Delta h \rho g \qquad \text{... (i)}$$

According to continuity equation

$$S_1 v_1 = S_2 v_2 \quad \text{or} \quad \frac{v_2}{v_1} = \frac{S_1}{S_2} \qquad \text{... (ii)}$$

Applying Bernoulli's theorem

$$\frac{P_1}{\rho} + \frac{1}{2}v_1^2 = \frac{P_2}{\rho} + \frac{1}{2}v_2^2$$

or

$$\frac{P_1 - P_2}{\rho} = \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \frac{v_1^2}{2}\left[\left(\frac{v_2}{v_1}\right)^2 - 1\right] \qquad \text{... (iii)}$$

Substituting the value of $(P_1 - P_2)$ from equation (i) and $\left(\frac{v_2}{v_1}\right)$ from equation (ii) in equation (iii), we get

$$\frac{\Delta h \rho g}{\rho} = \frac{v_1^2}{2}\left[\left(\frac{S_1}{S_2}\right)^2 - 1\right] \text{ or } \Delta h g = \frac{v_1^2}{2}\left[\frac{S_1^2 - S_2^2}{S_2^2}\right] \text{ or } v_1 = \sqrt{\left[\frac{2\Delta h g S_2^2}{(S_1^2 - S_2^2)}\right]} = S_2\sqrt{\left[\frac{2\Delta h g}{(S_1^2 - S_2^2)}\right]}$$

Now, $Q = S_1 v_1 = S_1 S_2 \sqrt{\left[\frac{2\Delta h g}{(S_1^2 - S_2^2)}\right]} = S_1 S_2 \sqrt{\left[\frac{2\Delta h g}{(S_2^2 - S_1^2)}\right]}$.

7. A vertical tube has diameter 0.016 m at its bottom end from which water flows out at the rate of 1.2 kg per minute. The pressure at the end is atmospheric pressure 0.7 m of mercury. If the diameter of the tube is 0.004 m at a height of 0.3 m from the bottom end, find the pressure there.

**Soln.** Mass of water flowing in one second = 1.2/60 = 0.02 kg.

Let $v_1$ be the velocity of flow of water at the bottom, then the mass of the water flowing per second $(\pi r^2) v_1 \rho$.

$$\therefore \quad (\pi r^2) v_1 \rho = 0.02$$

Here, $r_1 = \frac{0.006}{2} = 0.003$ m and $\rho = 10^3$ kg/m$^3$

$$\therefore \quad \{3.14 \times (0.003)^2\} v_1 10^3 = 0.02 \text{ or } v_1 = \frac{0.02}{3.14 \times (0.003)^2 \times 10^3} = 0.7074 \text{ m/sec.}$$

Let $v_2$ be the velocity of flow of water at the point where the radius of the tube is 0.004/2, i.e., 0.002 m. The mass of the water flowing per second is $\pi(0.002)^2 v_2 \times 10^3$. As the mass of the water flowing through any point in one second is the same, hence $\pi(0.002)^2 v_2 10^3 = 0.02$.

$$\therefore \quad v_2 = \frac{0.02}{3.14 \times (0.002)^2 \times 10^3} = 1.592 \text{ m/sec.}$$

Applying Bernoulli's theorem, we have $P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$.

Here, $P_1 = 0.76$ m of mercury

$= 0.76 \times (13.6 \times 10^3) \times 9.8$ N/m$^2$

$P_2 = h$ m of mercury

$$= h \times (13.6 \times 10^3) \times 9.8 \text{ N/m}^3$$

$$(h_2 - h_1) = 0.3 \text{ m},$$

$$v_1 = 0.7074 \text{ m/sec and } v_2 = 1.592 \text{ m/sec}.$$

Now, $$\frac{P_1 - P_2}{\rho} = \frac{1}{2}(v_2^2 - v_1^2) + g(h_2 - h_1)$$

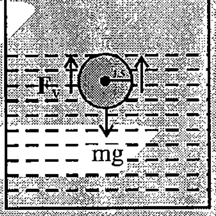
or $$\frac{(0.76 - h)\, 13.6 \times 10^3 \times 9.8}{10^3} = \frac{1}{2}\left[(1.592)^2 - (0.70742)^2\right] + 0.3 \times 9.8$$

$$0.76 - h = \frac{3.957}{13.6 \times 9.8} = 0.0297$$

$\therefore$ $h = 0.76 - 0.0297 =$ **0.7303 m of mercury.**

8. A steel ball of diameter d= 3.0mm sinking with zero initial velocity in olive oil whose viscosity $\eta = 0.90$P. How soon after the beging of motion will the velocity of the ball differ from the steady state velocity be n = 1.0 % ?

**Soln.** According to the question. The equation of motion.



$$m\frac{dV}{dt} = mg - 6\pi\eta rv$$

$\Rightarrow$ $$\frac{dV}{dt} + \frac{6\pi\eta r}{m} v = g$$

$\Rightarrow$ $$\frac{dV}{dt} + kV = g \quad , k = \frac{6\pi\eta r}{m}$$

Or, $$e^{kt}\frac{dv}{dt} + ke^{kt}v = ge^{kt};\ v\,e^{kt} = \frac{g}{k}e^{kt} + C;\ v = \frac{g}{k} + C\,e^{-kt} \text{ (where C = constant)}$$

Since v = 0 for t = 0, $0 = \frac{g}{k} + C,\ C = \frac{-g}{k}$

Thus, $$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

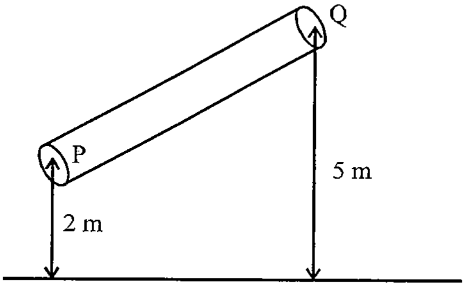
The steady stete velocity is $\frac{g}{k}$

v differs from $\frac{g}{k}$ by 'n' where $e^{-kt} = n$ , $t = \frac{1}{k}\ln n$

$$\frac{1}{k} = -\frac{\frac{4\Pi}{3}r^3\rho}{6\Pi\eta r} = \frac{-4r^2\rho}{18\eta};\ \frac{1}{k} = -\frac{d^2\rho}{18\eta}$$ ( By neglecting buoyncy in olive oil).

Now one can get eavily the value of 't' by potting the values of d, $\rho$ $\alpha$ $\eta$.

9. A non-viscous liquid of constant density $1000 \mathrm{kg}/\mathrm{m}^3$ flows in a streamline motion along a tube of variable cross section. The tube is kept inclined in a plane as shown in the fig. The area of crossection of the tube of two points at height of 2meter & 5 meter are respeetrelly $4\times10^{-3}\mathrm{m}^2$ & $8\times10^{-3}\mathrm{m}^2$. The velocity of liquid at point 'P' is 1m/se. Find the work done per unit volume by the pressure and the gravity forces as the fluid flows from point p to Q



**Soln.** As gravitational find is conserrative i.e. $W = -U$

Or, $$\left(\frac{dW}{dV}\right)_g = \frac{dU}{dV} = -\frac{mg(h_2-h_1)}{V} = -\rho g(h_2-h_1)$$

So work done by the force of gravity per unit volume

$$\left(\frac{dW}{dV}\right)_g = -\rho g(h_1-h_2) = -10^3\times9.8(5-2) = -2094\times10^4\frac{3}{m^3} \qquad ...(1)$$

Now in case of aideal fluid motion by consentration of mass i.e.,

$$\left(\frac{dm}{dt}\right)_1 = \left(\frac{dm}{dt}\right)_2$$

i.e., $$(\rho AV)_1 = (\rho AV)_2$$

Or, $$A_1V_1 = A_2V_2 = (B = \text{conet})$$

So $$V_2 = \frac{A_1V_1}{A_2} = \frac{4\times10^{-3}\times1}{8\times10^{-3}} = \frac{1}{2}\,\mathrm{m/se} \qquad ...(2)$$

Now as work done unit volume by pressure,

$$\left(\frac{dW}{dV}\right)_P = \frac{PdV}{dV} = P = P_2 - P_1$$

But by Bernoull is theorm

$$P_1 + \rho gh_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho gh_2 + \frac{1}{2}\rho v_2^2$$

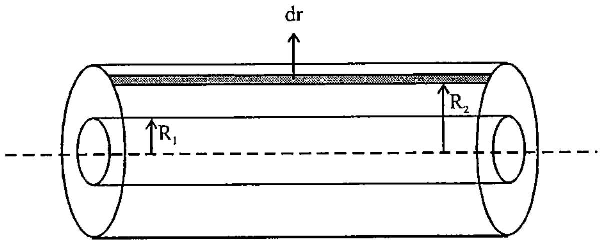
$$\Rightarrow \quad \left(\frac{dw}{dv}\right)_P = P_1 - P_2 = \rho g(h_2-h_1) + \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

Which in light of equation (1) & (2)

$$\left(\frac{dW}{dV}\right)_P = 2.94\times10^4 + \frac{1}{2}10^3\left[(0.5)^2 - 1^2\right] = 29025\ \mathrm{J}$$

10. Along Cylindrical radius $R_1$ is displaced along its axis with a containt velocity $V_0$ inside a stationary co-oxial cylinder of Radius. The space between the cylinder s is filled with the viscous liquids are function of the distance r from the axis of the cylinders. The flow is laminar

**Soln.**



Let's consider a co-oxial cylinder of radius 'r' & thickness dr, then force of friction or riscous force on this general Layer

$$F = 2\pi r\ell\eta\frac{dv}{dr}$$

This force must be constent form Layer to layer so that study motion may be possible

Or, $$\frac{fdr}{r} = 2\pi\ell\eta dv \quad ...(1)$$

On Intergrating,

$$f\int_{R_2}\frac{dr}{r} = 2\pi\ell\eta\int_0^v dv;\ f\ln\left(\frac{r}{R_2}\right) = 2\pi\ell\eta V \quad ...(2)$$

Putting, $r = R_1$ we get

$$f\ln\left(\frac{R_1}{R_2}\right) = 2\pi\ell\eta v_0 \quad ...(3)$$

form equation (2) & (3) we get

$$\boxed{v = v_0\frac{\ell n(r/R_2)}{\ell n(R_1/R_2)}}$$

**Note:** The force f is supplied by the agency which tries to carry the inner cylinder with velocity $V_0$.

11. When a sphere of radius $r_1 = 1.2\text{ mm}$ moves in glycerin the laminar flow is observed if the velocity of the sphere dues not exceed $V_1 = 23\text{ cm/sec}$. At what minimum velocity $V_2$ of a sphere of radius $r_2 = 5.5\text{ cm}$ will the flow in water because turbulent? the viscositives of glycerin and water are equal to $\eta_1 = 13.9\rho$ $\propto$ $\eta_2 = 0.071\rho$ respectively

**Soln.** We know that reynolais number for turbuient flow is greater than that of laminor flow

$$(Re)\ell = \frac{2\rho_1V_1r_1}{\eta_1} \qquad \left(\therefore = \frac{\rho vd}{\eta}\right)$$

And, $$(Re)_t = \frac{2\rho_2V_2r_2}{\eta_2} \qquad \text{But, } (Re)_t \geq (Re)\ell$$

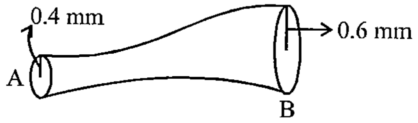
Then, $(V_2)_{min} = \dfrac{S_1 V_1 r_1 \eta_2}{S_2 r_2 \eta_1}$

By putting thye values weget

$$\boxed{(V_2)_{min} = 5\mu / se}$$

12. A tube as shown i fig. consisting of two values at its two ends. At end 'A' the the fluid having riscosity 9.1P exert a force of 1.5 N and gets out at another end 'B' by a force one the value measured as 0.6N. If the smaller and larger radii of the tube is given by 0.4 mm and 0.6 mm respectvely. Find out the flow rate at an average point inside the tube.

**Soln.**



Given that the forece at end 'A'

$$F_A = 1.5\ N.$$

Area of 'A' $= \pi r^2 = \pi(4)^2 = 16\pi\ cm^2$

13. The gross-seetional radius of a pipeline decrases gradually as $r = r_0 e^{-ax}$, where $\alpha = 0.50 m^{-1}$, x is the distance from the pipeline inlet. Find the ratio of Reynold number for two cross-see times seperated by $\Delta x = 3.2$ m.

**Soln.** we know that Reynold's number (Re) is defined as $Re = \dfrac{\rho V \ell}{\eta}$, where v is the velocity $\ell$ is characteristic length and $\eta$ the coeffient of viscosity. In this case of circular cross section the charectiristic length is the diameter of cross- section 'd' and V is taken as average velocity of flow of liquied,

Now Re, (Reynold os number at x, from the pipe end) $= \dfrac{\rho d_1 V_1}{m\eta}$ where $V_1$ is the velocity at distance x,

And similarly, $Re_2 = \dfrac{\rho d_2 V_2}{\eta}$ so $\dfrac{Re_1}{Re_2} = \dfrac{d_1 V_1}{d_2 V_2}$

From equation of continuity, $A_1 V_1 = A_2 V_2$

Or, $\pi r_1^2 V_1 = \pi r_2^2 V_2$

Or, $d_1 V_1 r_1 = d_2 V_2 r_2$ ; $\dfrac{d_1 V_1}{d_2 V_2} = \dfrac{r_2}{r_1} = \dfrac{r_0 e^{-ax_2}}{r_0 e^{-ax_1}} = e^{-a\Delta x}$ $(\text{as } x_2 - x_1 = \Delta x)$

Thus, $\dfrac{Re_2}{Re_1} = e^{a\Delta x} = 5$

14. A 3.6 m long vertical pipe resonates with a source of ferquency 212.5Hz when water level is at a certain heights is the pipe. Find the heights of water level (from the botton o the pipe) at which resonance occur. Neglet end correction. New the pipe is fvies to a heigher $H(\approx 3.6\text{m})$. A small hole is drilled very close to its bottom and water is allowed to leak. Obtain an expression for the rate of fall of water leves in the pipe as a function of H. If the radii of the pipe and hole are $2\times10^{-2}\,\text{m}$ and $1\times10^{-3}\,\text{m}$ respectivelly, calculates the time interval between the occurance of first two resonances. Speed of sound is air is 340 m/s and $g = 10\,\text{m}/\text{s}^2$.

**Soln.** For resonance to occur in a closed pipe,

$$\ell = \frac{(2n-1)\lambda}{4} \quad \text{where } n = 1, 2, 3........$$

Here, $$\lambda = \frac{V}{\eta} = \frac{340}{212.5} = 1.6\text{ m}$$

So resonce occuer for length of air column 0.4m, 1.2m, 2.0 m, 2.8m and 3.6m, and corresponding heighet of water column from bottom.

$$= 3.2\text{m},\ 2.4\text{m},\ 1.6\text{m},\ 0.8\text{m, and } 0\text{m}$$

The velocity of eff;ux, when height of water level is h above bottom $= \sqrt{2gh}$

Applying continuity equation

$$A\left(-\frac{dh}{dt}\right) = a\sqrt{2gh}$$

where a and a are area of cross- section of hole and pipe respectively

$$dt = -\frac{A}{a\sqrt{2gh}}dh$$

So, the time in which the height of water column changes from 3.2 m to 2.4 m is time interva between first to resonances,

$$t = -\frac{A}{a}\int_{3.2}^{2.4}\frac{1}{\sqrt{2gh}}dh = -\frac{A}{a\sqrt{2g}}2\left|h^{1/2}\right|_{3.2}^{2.4}$$

$$= \frac{\pi\left(3\times10^{-2}\right)^2\times2}{\pi\left(1\times10^{-3}\right)\sqrt{2\times10}}\left[\sqrt{3.2}-\sqrt{2.4}\right] = 42.87\text{ s}$$

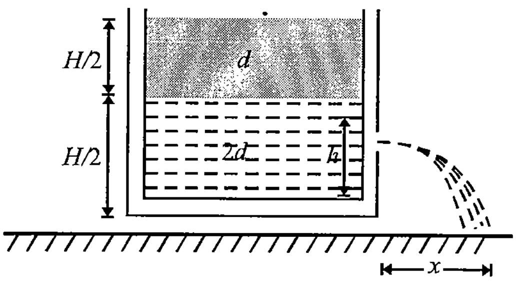
15. A container of large uniform cross- section area A resting on a horizontal surface holds two immiscible non-viscoss and incompressible liquids of densities d are 2d each of height $(H/2)$ as shown is fig. 1.The lower density liquid is ofen to the atoms phere having prssures Po (a) A homogeneous solid cylinder of length $L(L < H/2)$, cross-sectional area (A/s) is Immrsed such that it floats with uits axis vertical a at the liquid-liquid interface with length (L/4) in the densder liquid. Dtermine (i) the density of solid and (ii) the total prssure at the bottom of the contaier (b) the cylinder is removed and original arrangement is restored. A ting hole of area S(S<<A) is punched on the vertical side of the container at a hight $h(h < H/2)$.

Determine (i) the initial speed of efflux of the liquid of the hole

(ii) The horizontal distance x travelled by the liquied inttally and

(iii) The hight $h_m$ at which the hole should be punched so that the liquid travels the maximum distance $x_m$ ......

Also calculate $x_m$



**Soln.** (a) (i) As for floating, w = Th

$$VPg = V_1 d_1 g + V_2 d_2 g$$

Or, $$L\left(\frac{A}{5}\right)\rho = \left(\frac{3}{4}L\right)\left(\frac{A}{5}\right)d + \left(\frac{1}{4}L\right)\left(\frac{A}{5}\right)2d$$

i.e., $$\rho = \frac{3}{4}d + \frac{2}{4}d = \frac{5}{4}d$$

(ii) Total pressure = $P_0$ + (weight of liquid + weight of solid) |A

$$P = P_0 + \frac{H}{2}dg + \frac{H}{2}2dg + \frac{5}{4}d \times \left(\frac{A}{5} \times L\right) \times g \times \frac{1}{A}$$

$$P = P_0 + \frac{3}{2}Hdg + \frac{1}{4}Ldg = P_0 + \frac{1}{4}(6H + L)dg$$

(b) (i) By Bernoulli 's theoram for a point first inside and outside the hole

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$$

i.e., $$P_0 + \frac{H}{2}dg + \left(\frac{H}{2} - h\right)2dg = P_0 \frac{1}{2}(2d)V^2$$

$$g(3H - 4h) = 2V^2 \quad V = \sqrt{(g/2)(3H - 4h)}$$

(ii) As at the hole vertical velocity of liquid is zero to time taken by it to reach the ground,

$$t = \sqrt{(2h/g)}$$

So that $$x = vt + \sqrt{\frac{g}{2}(3H - 4h)}\sqrt{\frac{2h}{g}} = \sqrt{h(3H - 4h)}$$

(iii) For x to be maximum, $x^2$ must be maximum i.e.,

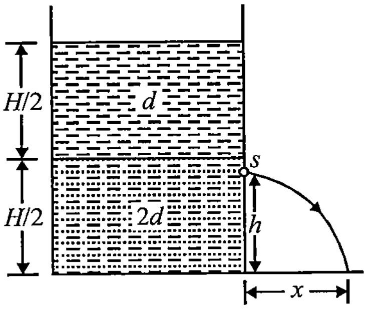
$$\frac{d}{dh}\left(x^2\right) = 0$$

Or, $$\frac{d}{dh}\left(3Hh - 4n^2\right) = 0$$

$3H - 8h = 0$, i.e., $h = (3/8)^H$

And $$x_{\max} = \sqrt{\frac{3H}{8}\left(3H - \frac{3}{2}H\right)} = \frac{3}{4}H$$

16. A container of large uniform cross-sectional area $A$ resting on a horizontal surface holds two immiscible, non-viscous and incompressible liquids of densities $d$ and $2d$, each of height $H/2$ as shown. The lower density liquid is open to the atmosphere having pressure $P_0$. A tiny hole of area $s\ (s \ll A)$ is punched on the vertical side of the container at a height $h(h < H/2)$. Determine :



(i) the initial speed of efflux of the liquid at the hole.

(ii) the horizontal distance distance $x$ travelled by the liquid initially, and

(iii) the height $h_m$ at which the hole should be punched so that the liquid travels the maximum distance $x_m$ initially. Also calculate $x_m$.

(Neglect the air resistance in these calculations.

**Soln.** (i) Here the effective height of orifice can not be taken as $(H-h)$ due to the pressure of two liquids. Using Bernoulli's theorem, just inside and just outside the orifice the orifice, we have

$$P_0 + \frac{H}{2}dg + \left(\frac{H}{2} - h\right)(2d)g = P_0 + \frac{1}{2}(2d)v^2$$

where $v$ is the velocity of efflux

$$\therefore \quad \frac{H}{2}dg + \frac{H}{2}(2d)g - h(2d)g = \frac{1}{2}(2d)v^2$$

$$\text{or} \quad \frac{3}{2}Hg - 2hg = v^2$$

$$\text{or} \quad v = \sqrt{\left[\frac{g}{2}(3H-4h)\right]} \qquad \text{...(i)}$$

(ii) Let $t$ be the time taken by the liquid to hit the ground. Then

$$h = \frac{1}{2}gt^2 \ \text{ or } \ t = \sqrt{\left(\frac{2h}{g}\right)}$$

The horizontal range $x = v \times t$

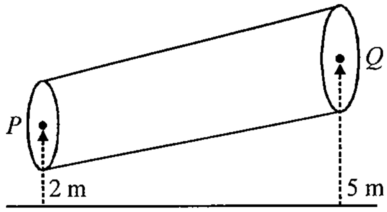
$$\therefore \quad x = \sqrt{\left[\frac{g}{2}(3H-4h)\right]} \times \sqrt{\left(\frac{2h}{g}\right)} = \sqrt{[(3H-4h)h]} \qquad \text{...(ii)}$$

(iii) To find out the maximum height $h_m$, we differentiate $x$ with respect to $h$ and equate it to zero for maximum $h$. Hence, $\frac{dx}{dh} = \frac{d}{dh}\sqrt{[(3H-4h)h]} = 0$.

$$\therefore \quad 3H - 8h = 0 \ \text{ or } \ h = \frac{3}{8}H \qquad \text{...(iii)}$$

$$\therefore \quad x_m = \sqrt{\left[\left(3H - \frac{4\times 3}{8}H\right)\frac{3}{8}H\right]} = \frac{3H}{4}. \qquad \text{...(iv)}$$

17. A nonviscous liquid of constant density 1000 kg/m$^3$ flows in a streamline motion along a tube of variable cross section. The tube is kept inclined in the vertical plane as shown in figure. The area of cross-section of the tube at two points $P$ and $Q$ at height of 2 meters and 5 meters are respectively $4 \times 10^{-3}$ m$^2$ and $8 \times 10^{-3}$ m$^2$. The velocity of the liquid at point $P$ is 1 m/s. Find the workdone per unit volume by the pressure and the gravity forces as the liquid flows from point $P$ to $Q$.



**Soln.** The workdone per unit volume by the force of gravity,

$$W_{g/vol.} = \rho g \Delta h$$

$$= 10^3 \times 9.8 \times 3 = \mathbf{2.94 \times 10^4\ J/m^3}.$$

From the equation of continuity, $A_1 v_1 = A_2 v_2$ or $v_2 = \dfrac{A_1 v_1}{A_2}$

$$\therefore \quad v_2 = \frac{4 \times 10^{-3} \times 1}{8 \times 10^{-3}} = 0.5 \text{ m/s}.$$

Now, increment of kinetic energy of fluid per unit volume.

$$= \frac{1}{2}\rho(v_2^2 - v_1^2) = \left(\frac{1}{2} \times 10^3\right)\left[\frac{1}{4} - 1\right] = -3.75 \times 10^2 \text{ J/m}^3.$$

We know that net energy of the fluid per unit volume, in a stream line flow remains the same and hence the net workdone by pressure per unit volume of the fluid is given by

$$W_{p/vol.} = (2.94 \times 10^4 - 3.75 \times 10^2) = \mathbf{29025\ J/m^3}.$$